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Asymmetric Conflict

Weakest Link against Best Shot

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The authors study conflict on multiple fronts. A defending player needs to successfully defend all fronts, and an attacker needs to win at only one. Multiple fronts result in a considerable disadvantage for the defending player, and even if there is a defense advantage at each of them, the payoff of the defending player is zero if the number of fronts is large. With some positive probability, in the equilibrium defending players surrender without expending effort.

Keywords: *weakest link; best shot; multiple fronts; contest; asymmetric conflict*

Introduction

A military player is sometimes forced to fight on two or more fronts and split up his forces in an effort to defend each of them. An example of such a two-front war is World War I, in which Germany had to fight against France in the West and against Russia in the East. Otto von Bismarck had used much of his diplomatic effort in trying to isolate France and prevent Germany from being encircled and ending up in a multifront war by devising a policy of alliances after the German-French War of 1870.[1] After he was dismissed in 1890, his policy was discontinued. When the threat of a two-front war became imminent, the Germans tried to deal with it by developing, and eventually trying to implement, what is known as the Schlieffen Plan. This was a blitzkrieg against France that should ideally have been finished by the time the Russian army was fully mobilized. When this plan failed, Germany ended up in a two-front war.

There are many other situations in which one player faces threats on multiple fronts and where success is ensured only if he is successful on all of them.

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Guarding and defending a city that has many gates is a further example in the military context that depicts the nature of the problem well. Supply chains, chains of command, or information that can be intercepted or interrupted at several links exhibit similar, multiple vulnerability. Intercepting such chains at one of the links may be sufficient to make all of them useless. This also applies to the context of business and administration. If an organization is vulnerable along several dimensions, it may survive, or be successful, only if it is protected on all dimensions. To sabotage production successfully, it may be enough to succeed in sabotaging one component of a product, or in interrupting the production chain at one point. A similar logic applies in politics. A politician may have done, or said, many things in the past that make him vulnerable in several dimensions. To sabotage his campaign or to destroy his reputation, it may be enough to be successful in one dimension. Finally, consider a human body that is attacked by a virus. Several cells will typically be attacked by a number of viruses. For the attack to be successful in a first round of infection, it may be enough for one of these cells to be invaded successfully. All these examples have in common that the defending player is exposed to an attack at several points, and that a defeat at any of these points implies a final defeat for the defending player.[2]

The multifront war with the weakest-link/best-shot property puts the defending player at a strong strategic disadvantage. As we show in the next section, if the number of fronts is large, the defending player essentially becomes indifferent to whether to surrender or fight, and his payoff is reduced to the one he receives from surrender. In the equilibrium, the defending player surrenders with positive probability, and also fights back with positive probability. The probability of fighting declines as the number of fronts increases. These results also hold even if the defending player has a local advantage of defense at each single front.[3]

The article is related most closely to analyses of conflict with multiple, simultaneous battles. One part of this literature studies versions of the Colonel Blotto game, in which each of two players is endowed with a given amount of resources and has to decide how to use these resources on several alternative fronts. This game has a long tradition in the literature on military conflict (e.g., Blackett 1954), on marketing (e.g., Friedman 1958), and on electoral competition (e.g., Snyder 1989; Myerson 1993; Laslier and Picard 2002). The literature is also surveyed in recent contributions by Kvasov (forthcoming), Robson (2005), and Roberson (2006), who solve a very general class of Blotto games. Kovenock and Roberson (2006) consider multifront conflict in the context of terrorism. They solve a very general set of problems that encompasses the weakest-link/best-shot problem considered in our article. However, they apply a different contest technology in which contest success is discontinuous in effort and attribute the victory in a battle with probability 1 to the player who expends the greater effort. This yields equilibrium outcomes that are structurally different from ours.

The general idea of asymmetry, according to which one player is victorious if he wins a single battle while his adversary must be successful along all battles to take home final victory, is related to Hirshleifer's (1983) discussion of the noncooperative provision of public goods and how the equilibrium outcome depends on how the different individual contributions determine the quantity or quality of the public good that results. He looks at two extreme cases: in one case, the minimum contribution determines the aggregate level, due to strict complementarity of individual contributions. He calls this "weakest link," referring to a situation in which the weakest element of a chain determines the maximum force that the chain can sustain. The second case is called "best shot," in which the highest—or most effective—contribution from a set of contributors characterizes the aggregate level of the public good. Military applications come into mind when thinking about these technologies. The weakest link is typically associated with defense problems, like defending the various gates of a city against intruders; important dispatches that can be intercepted at several points; or the coastline of France, which had to be defended against the landing of the Allied troops in the later part of World War II. The best shot, in turn, is most naturally associated with the respective problems of attack. Accordingly, weakest link and best shot often emerge jointly and for the same problem, and it is therefore interesting to study the interaction between two rivals or rival groups, one facing a weakest-link problem and the other facing a best-shot problem.

The Analytics of Multiple Battlefields

Consider two players A and D. Player A attacks a country and player D defends it. The country is protected by mountains and other natural obstacles, but there are n points or possible fronts at which an attack can be made and has some potential for being successful. Player A decides at which points to attack and can try to attack any number of these fronts. A battle contest takes place on each such front i, in which A expends an amount $x_i \in [0, K]$ of effort. His unit cost of effort is $a > 0$, which can be larger or smaller than 1. The defending player D expends an amount $y_i \in [0, K]$ of effort. His unit cost is normalized to unity. Here K is assumed to be sufficiently large to be never binding. The probabilities for D and A winning the battle at front i are denoted p_D^i and $p_A^i = 1 - p_D^i$. These are functions of x_i and y_i that will be characterized further below.

Formalizing the best-shot/weakest-link property, player A wins the overall conflict if he wins at least at one front, whereas player D only wins if he successfully defends his territory along all n fronts. A's and D's valuations of winning are v_A and v_D, respectively. A's payoff is

$$\pi_A = \left(1 - \prod_{i=1}^{n} p_D^i(x_i, y_i)\right) v_A - \sum_{i=1}^{n} a x_i. \qquad (1)$$

Player A wins with a probability that is equal to *1* minus the probability that D wins along all battle fronts. He has to bear his cost of effort which is the sum of the efforts in the different contests. The payoff of D is equal to his probability of winning all battle fronts times the valuation of winning, minus the sum of efforts y_i:

$$\pi_D = \prod_{i=1}^{n} p_D^i(x_i, y_i)\nu_D - \sum_{i=1}^{n} y_i. \tag{2}$$

Note that ν_A is the prize that A attributes to winning *at least one battle*, and ν_D is the prize that D attributes to winning *all battles*.[4] These prizes may, but need not be, identical.[5] For instance, the gain an army may expect from successfully conquering and plundering a city is likely to be less than what the citizens lose in this case, particularly if plundering goes along with vandalism and violence.

Player D wins the conflict at front i with probability

$$p_D^i(x_i, y_i) = \begin{cases} \dfrac{y_i}{x_i + y_i} & \text{if } x_i > 0 \\ 1 \text{ otherwise,} & \end{cases} \tag{3}$$

and $p_A^i = 1 - p_D^i$. The contest success function (3) is well known from many areas of application. The win probability of a contestant in (3) is determined like in a lottery in which A and D both purchase lottery tickets that all go into one urn, one ticket being drawn randomly and determining the winner.[6] Note that $p_D^i(0, 0) = 1$ describes that, if no attack occurs on front i, the defending player wins on this front, even if he does not make any positive defense effort. While this assumption is plausible, it is a tie-breaking rule that is useful in what follows, but it is not crucial for the qualitative results. The possibly different unit cost of efforts for defense and attack allow for a local asymmetry between attack and defense at each front, taking into account an important claim in the theory of military strategy that suggests that there is a local advantage of defense (Clausewitz 1832/1976, 84). For instance, for the 1970s, U.S. Minister of Defense James R. Schlesinger (1975, III-15) suggested that one unit of resources expended by the defending player may counterbalance three units of resources expended by the attacker in conventional warfare. Given that the unit cost of defense effort is normalized to unity, this would be described here by a unit cost $a = 3$ for the attacking player.

An equilibrium in pure strategies is described by a pair of feasible effort vectors $\mathbf{x}^* = (x_1^*, \ldots, x_n^*)$ and $\mathbf{y}^* = (y_1^*, \ldots, y_n^*)$ such that $\pi_A(\mathbf{x}^*, \mathbf{y}^*) \geq \pi_A(\mathbf{x}, \mathbf{y}^*)$ for all feasible $\mathbf{x}$, and $\pi_D(\mathbf{x}^*, \mathbf{y}^*) \geq \pi_D(\mathbf{x}^*, \mathbf{y})$ for all feasible $\mathbf{y}$. The equilibrium may be in mixed strategies, which are defined naturally, that is, as cumulative distribution functions of feasible pure strategies.

We now note the following property that will simplify the equilibrium considerations.

Lemma 1: If $y_i = y_j > 0$ and $x_i \neq x_j$ for $i, j \in \{1, 2, \ldots, n\}$, then A can weakly increase his payoff by choosing $\hat{x}_i = \hat{x}_j = \frac{x_i + x_j}{2}$. Also, if $x_i = x_j > 0$ and $y_i \neq y_j$ for some $i, j \in \{1, 2, \ldots, n\}$, then D can weakly increase his payoff by choosing $\hat{y}_i = \hat{y}_j = \frac{y_i + y_j}{2}$.

Intuitively, suppose a city has n identical gates. An attacker sends an equal number of soldiers to gates 1 and 2. Then the defending player should optimally not send more troops to one gate than to the other. Lemma 1 helps reduce the complexity of the $n-$dimensional effort choice of each contestant to a one-dimensional problem of choosing effort levels that are uniform across the different fronts. The lemma rests on the assumption that contest success is governed by the same rules along all fronts. We prove the following proposition in the appendix:

Proposition 1. (i) If $a\nu_D \leq (n-1)\nu_A$, then an equilibrium exists and is characterized by player A choosing $\mathbf{x} = (x^*, x^*, \ldots, x^*)$ with

$$x^* = \frac{(n-1)^{n-1}}{n^{n+1}} v_D, \tag{4}$$

and player D choosing $\mathbf{y}^* = (y^*, \ldots, y^*)$ with

$$y^* = \begin{cases} \frac{(n-1)^n}{n^{n+1}} v_D & \text{with probability} \quad q^* = \frac{v_D}{v_A/a} \frac{1}{n-1} \\ 0 & \text{with probability} \quad 1 - q^* \end{cases} \tag{5}$$

Player A's expected payoff is $\pi_A^* = v_A - 2av_D \frac{(n-1)^{n-1}}{n^n}$, and player D's expected payoff is $\pi_D^* = 0$.

(ii) If $a\nu_D > (n-1)\nu_A$ holds, then an equilibrium exists and is characterized by

$$x^* = \frac{\left(\frac{v_A}{a}\right)^2 v_D^n}{\left(\frac{v_A}{a} + v_D\right)^{n+1}} \text{ and } y^* = \frac{\frac{v_A}{a} v_D^{n+1}}{\left(\frac{v_A}{a} + v_D\right)^{n+1}} \tag{6}$$

and by $q^* = 1$. Both players have strictly positive payoffs in this equilibrium.

Three factors determine whether the equilibrium is characterized by (i) or by (ii). For any size of the advantage of defense, or for the ratio of the valuations of winning, the mixed strategy equilibrium in (i) prevails if the number of battle fronts becomes sufficiently large. Intuitively, the battle victories at the different fronts are stochastically independent events. The attacker has a positive chance of winning at each of these fronts if he expends positive effort along them, even if his effort is quite small. Hence, if he allocates some effort to each front, it becomes very likely that the attacker will win at least one battle contest, even if this effort is not high, and this is sufficient for him to win the overall game. The attacking player benefits from an increase in the number of fronts: each additional front gives him an additional chance of victory. In contrast, the defending player does not like additional fronts. Defending a particular front is not particularly rewarding if there are many fronts, as the attacker is in any case likely to be successful on one of the other fronts. In the case of an increase in the number of fronts, to keep the probability by which he is successful along all fronts constant, the defending player would have to expend more effort at each of the fronts and increase the win probability at each front, as his win probability is the product of the win probabilities for each single front. This is very costly, and increasingly so the more fronts there are.

The nature of the mixed strategy equilibrium in (i) can also be explained in intuitive terms. Player D has zero payoff in this equilibrium. Zero payoff does not imply, however, that D expends zero effort along all fronts with probability 1 in the equilibrium. If $\mathbf{y} = 0$ always, the attacker would win for sure with an infinitesimally small effort. But given this infinitesimally small effort by A, the choice of $\mathbf{y} = 0$ would no longer be optimal for player D, as D would need only a small amount of effort to drive the win probability at each front up to close to 1. Hence, the defending player cannot simply give up with certainty. He can also not optimally choose a given amount of effort with certainty: the strictly positive effort levels that are locally best responses to each other would imply so much defense effort that the defender's payoff would become negative, so he would be better off abstaining from any defense effort. Accordingly, the defending player randomizes between zero effort and a strictly positive effort $\mathbf{y}^*$. With an increase in the probability q by which D expends this positive effort $\mathbf{y}^*$, it becomes worthwhile for player A to also expend higher effort $\mathbf{x}^*$. The equilibrium q causes the mutually optimal replies $\mathbf{y}^*$ and $\mathbf{x}^*$ to be just low enough to make the defending player indifferent between giving up and choosing the optimal positive effort $\mathbf{y}^*$.

Note that the probability with which the defending player chooses zero effort increases in the number of fronts, and in the ratio v_A/v_D of players' valuations.

If there are only a few fronts, if the valuation of winning is sufficiently higher for the defending player than for the attacking player, or if the local advantage of defense is sufficiently high, $av_D > (n-1)v_A$ may hold and the equilibrium is as in (ii). In this case, it is worthwhile for the defending player to consistently, and always, defend all fronts. Both players expend mutually optimally chosen efforts uniformly along all fronts.

Note that $av_D > (n-1)v_A$ is always fulfilled if there is only one front, as in the standard Tullock (1980) contest. This shows that adding an additional front qualitatively changes the nature of the problem.

Historically it can be seen that cities that came under siege sometimes surrendered, and sometimes fought and tried to succeed.[7] This fits with the equilibrium properties in the mixed strategy equilibrium in (i). Some of these cities may fight, even though they are essentially indifferent between early surrender and likely surrender after considerable resistance. The main result suggests that it is in the interest of a defending player to have only few weak points at which he can be attacked. The reduction in the number of these points may be more important than making it easier to defend each of these points. The attacker has the opposite interest and would like to be able to attack on more points. This may apply to defense structures in military conflict, but also to nonmilitary applications. It may call for institutions that are designed in a way that makes them vulnerable only at a few points.

The result also has implications for the formation of alliances. In the context of international military conflict, geography matters and a common battle front typically requires an immediate neighborhood between attacking and defending countries. Alliances may change the number of points at which the defending side can be attacked. Additional fronts at which a country can be attacked may become feasible through an alliance.[8] The weakest-link/best-shot nature of conflict may therefore point at a new aspect of alliances and possibly give them additional strategic value. Alliances may also change the degree of complementarity between the contest outcomes along the set of fronts, however, making the policy conclusions less straightforward. In addition to considering alliances as changing the number of points of attack or defense, the types of equilibria that we have identified may also help to explain the composition of strategic alliances. Consider a union of countries that are considering admitting a new member and that initially the mixed strategy equilibrium (i) in Proposition 1 is being played; this new member will likely increase the number of points of vulnerability of the alliance but may well also change the value to the defending party of successful defense (v_D in the formal analysis). According to the results of Proposition 1, it would be in the interests of the defending player to admit the newcomer if the extra value of successful defense were such that the pure strategy equilibrium (ii) could be played. Forging this alliance facilitates a transition from a zero expected payoff for the defender to a positive one in equilibrium. A similar argument can be made for the attacker; here it may be of importance to ally oneself with those who will give a large increase in the value of successful attack so that the defending party is more likely to surrender immediately. A similar case can be made for the forging of strategic alliances in industry.

Proposition 1 allows only for a limited amount of asymmetry between attack and defense. To account for further asymmetries more formally, we may allow for differences in the unit cost that is required to generate effective efforts x_i and y_i among different fronts and between the two players. More specifically, let a_i and d_i be player A's and player D's cost per unit of effort at battle front i. If $a_i > d_i$, this describes a local advantage of defense at front i. For instance, to have the same win probability at front i given equation (3), both players need to provide the same effective effort: $x_i = y_i$. However, to provide this effort will cost player A who attacks a_i/d_i times what it costs the defending player D. In addition, if $a_i \neq a_j$ and/or $d_i \neq d_j$, this describes possible differences in the cost of defending or attacking at different fronts. Allowing for these asymmetries, the payoff functions can be redefined, replacing the cost term in equation (1) by $\Sigma_{i=1}^{n} a_i x_i$ and in equation (2) by $\Sigma_{i=1}^{n} d_i y_i$, respectively. This problem can generally not be reduced to a problem in which both players maximize along a single dimension.[9] As shown in the appendix, if there is an equilibrium similar to (i) in which the attacking player chooses some effort vector $(x_1^*, \ldots, x_n^*)$ and the defending player chooses $\mathbf{y} = 0$ with probability $(1 - q)$ and $(y_1^*, \ldots, y_n^*)$ with positive efforts with the remaining probability q, the equilibrium win probabilities that emerge from such asymmetries in those cases in which the defending player chooses the optimal positive defense effort $\mathbf{y}^*$ are $p_D^i(x_i^*, y_i^*) = a_i/(qd_i + a_i)$. A large effort cost of player A or a small effort cost of player D at a particular battle front i turns into a higher probability for the defending player winning this battle front in the equilibrium. Intuitively, consider, for instance, player D defending several town gates. Suppose it is more costly to defend gate 1 and more costly to attack gate 2. These asymmetries lead to an equilibrium in efforts in defense and attack that make it more likely that a successful invasion will occur at gate 1.

Discussion

A fundamental asymmetry between attack and defense comes into play if the defending player is vulnerable at several points and has to be successful in protecting all these points against the attacker who wins if he is successful at least at one such point. Using the language from military conflict, we denoted these points as fronts in a war, where an attacker wins the war if he wins at least on one front, and the defending player wins the war only if he successfully defends all fronts. We considered this problem using the most commonly used contest technology that has the property of globally decreasing returns, such that an attacker is willing to attack simultaneously at several such points.

A key result is that the defending player is willing to give up if the number of vulnerable points is sufficiently large, because the amount of resources needed to optimally react to an attack along all these possible fronts becomes so large that such effort is not better for this player than the option to expend zero effort. In the equilibrium, the defending player gives up with a certain probability but fights with some remaining probability. The probability of giving up is higher the larger the number of vulnerable points or fronts.

We allowed for local advantages of defense in each single battle front in the analysis. If these advantages are large, they expand the range in which the defending player does not give up and has a positive equilibrium payoff. However, for any given advantage of defense, there is always a finite number of battle fronts such that, if the number of battle fronts is larger or equal to this number, the defending player gives up. This suggests that the number of points of vulnerability, or the number of possible battle fronts, is an important strategic variable that is generally to the disadvantage of the defending player, who must be successful along all fronts.

This result has many applications. It may be relevant in the context of military conflict, but further examples are organizations that can be infiltrated or hurt by attacks in different units, communication chains that can be intercepted at several points, and so on. The key aspect in all these examples is the perfect complementarity of battle victories for the defending player and the perfect substitutability of battle victories for the attacking player, which places the defending player in a disadvantaged position. Defending players should try to minimize the number of fronts at which they can be attacked, and attackers may find it profitable to increase the number of points at which they can attack. As alliances may change the number of points of attack, this effect may increase the strategic value of alliances.

Appendix

Proof of Lemma 1. Consider π_A in equation (1). Let $y_i = y_j \equiv y > 0$ and $x_i \neq x_j$ for some $i, j \in \{1, \ldots, n\}$. Define $\Sigma_{k=1}^{n} a x_k \equiv \rho$. A's payoff can be written as

$$\pi_A = \left(1 - \frac{y}{x_i + y}\frac{y}{x_j + y}\prod_{k \neq i,j}^{n} \frac{y_k}{x_k + y_k}\right) v_A - \rho. \tag{7}$$

Define $\prod_{k \neq i,j}^{n} \frac{y_k}{x_{sk} + y_k} = \Psi$. If $\Psi = 0$, then the allocation of $x_i + x_j$ between battles i and j does not matter. If $\Psi > 0$ then π_A is strictly decreasing in $\frac{y}{x_i + y}\frac{y}{x_j + y}$. For a given sum of efforts $\hat{x}_i + \hat{x}_j = x_i + x_j$, the sum $\frac{y}{\hat{x}_i + y}\frac{y}{\hat{x}_j + y}$ is minimized for $\hat{x}_i = \hat{x}_j = \frac{x_i + x_j}{2}$.

The proof for the respective property for D's effort follows analogous lines.

Proof of Proposition 1. We show that the candidate equilibrium strategies for (i) are optimal replies to each other. Suppose first that D chooses $\mathbf{y} = (y^*, \ldots, y^*)$ with $y_i^* = \frac{(n-1)^n}{n^{n+1}} v_D)$ with probability q^* and $\mathbf{y} = 0$ with probability $1 - q^*$. Consider player A. Dividing the payoff function of player A by a yields

$$\frac{\pi_A}{a} = q^* \left(1 - \prod_{i=1}^{n} p_D^i(x_i, y^*)\right) \frac{v_A}{a} + (1 - q^*) \frac{v_A}{a} - \Sigma_{i=1}^{n} x_i. \tag{8}$$

The optimal $\mathbf{x}^*$ must maximize (8), with $p_D^i(x_i, y_i^*)$ defined in (3). Using Lemma 1, given $y_1^* = y_2^* = \cdots = y_n^* \equiv y^*$, the maximum has $x_i = x_j$ for all i, j. The maximization problem can be rewritten as

$$\frac{\pi_A}{a} = q^* \left(1 - \frac{(y^*)^n}{(x + y^*)^n}\right) \frac{v_A}{a} + (1 - q^*) \frac{v_A}{a} - nx \rightarrow \max_{x \in [0,K]}. \tag{9}$$

The first-order condition of this problem is

$$q^* \frac{(y^*)^n}{(x + y^*)^{n+1}} \frac{v_A}{a} - 1 = 0. \tag{10}$$

It can be checked that $x^* = \frac{(n-1)^{n-1}}{n^{n+1}} v_D$ as in (4) solves this equation for values $y^* = \frac{(n-1)^n}{n^{n+1}} v_D$ and $q^* = \frac{v_D}{v_A/a} \frac{1}{n-1}$ as in (5). Furthermore, $\pi_A = q^*(1 - \frac{(y^*)^n}{(x+y^*)^n}) v_A + (1 - q^*) v_A - nax$ is strictly concave in x for $y^* = \frac{(n-1)^n}{n^{n+1}} v_D$ and $q^* = \frac{v_D}{v_A/a} \frac{1}{n-1}$. To see this, note that

$$\frac{\partial^2 \pi_A}{(\partial x)^2} = -q \frac{y^n}{(x+y)^{n+2}} n(n+1) v_A < 0 \tag{11}$$

for $y > 0$ and $q > 0$. Finally, $\frac{\partial \pi_A(x, y^*, q^*)}{\partial x} > 0$ at $x = 0$, and, hence, $\pi_A(x^*, y^*, q^*) > 0$, which completes the proof that x^* is a globally optimal reply given q^* and y^*.

Consider next the optimal reply of D, given the equilibrium candidate choice by A. Player D chooses $\mathbf{y}$ to maximize

$$\pi_D = \prod_{i=1}^{n} p_D^i(x^*, y_i) v_D - \Sigma_{i=1}^n y_i. \tag{12}$$

Using again Lemma 1, given $x_1^* = x_2^* = \cdots = x_n^* \equiv x^*$, the maximum has $y_i = y_j$ for all i, j for all $i = 1, \ldots, n$. Therefore, the payoff function (2) can be rewritten as

$$\pi_D = \frac{y^n}{(x^* + y)^n} v_D - ny. \tag{13}$$

The first-order condition for an interior maximum is

$$\frac{x^* y^{n-1}}{(x^* + y)^{n+1}} v_D - 1 = 0. \tag{14}$$

By inserting x^* from (4), it can be checked that $y^* = \frac{(n-1)^n}{n^{n+1}} v_D$ solves this equation at $x^* = \frac{(n-1)^{n-1}}{n^{n+1}} v_D$. Furthermore, inserting y^* and x^* into π_D in (13) yields $\pi_D(y^*, x^*) = 0$. Accordingly, for $x = x^*$, player D is indifferent to whether to expend effort $\mathbf{y} = (y^*, \ldots, y^*)$, or $\mathbf{y} = 0$, or to randomize between these choices with a probability q^*.

It remains to be shown that $\mathbf{y} = \mathbf{y}^*$ and $\mathbf{y} = 0$ yield a global maximum of the player D's payoff for a given effort of A of $\mathbf{x} = (x^*, \ldots, x^*)$. To confirm this, note that $\pi_D = \frac{y^n}{(x^*+y)^n} v_D - ny$ is continuous on $y \in (0, K]$. Moreover,

$$\frac{\partial^2(\pi_D)}{(\partial y)^2} = \frac{ny^{n-2}}{(x+y)^n} \left(n - 1 - \frac{2yn}{(x+y)} + \frac{(n+1)y^2}{(x+y)^2} \right) v_D. \tag{15}$$

Accordingly, the sign of this expression is the same as the sign of $(n-1)(x+y)^2 - 2yn(x+y) + (n+1)y^2$, which is positive for strictly positive x and y, if, and only if, $(n-1)x > 2y$, or inserting x^*, if

$$y < \frac{1}{2}\frac{(n-1)^n}{n^{n+1}}v_D = \frac{y^*}{2}. \tag{16}$$

This shows that, for $x = x^*$, player D's payoff is strictly convex in the range $y \in (0, \frac{y^*}{2})$, and strictly concave for $y \in (\frac{y^*}{2}, K)$. Accordingly, the function has, at most, two local maxima, one being the local interior maximum at $y = y^*$, and the other at $y = 0$.

The equilibrium payoff of A follows from inserting the equilibrium values of effort and for q^* into (9) and simplifying. This payoff is strictly positive given $av_D \leq (n-1)v_A$, as a sufficient condition for this to hold becomes $\frac{n^n}{(n-1)^n} > 2$, and $\frac{n^n}{(n-1)^n}$ converges toward $e > 2$ for $n \to \infty$. The equilibrium payoff of D follows from the fact that $\mathbf{y} = 0$ is in the equilibrium support and yields $\pi_A = 0$.

Turn now to the case with $av_D > (n-1)v_A$. In this case (4) and (5) are the simultaneous solutions of the first-order conditions (10) and (14) for $q^* = 1$. Moreover, for $av_D > (n-1)v_A$, the payoff of both players A and B is strictly positive for these values, and the convexity/concavity properties of (9) and (13) hold unchanged. Hence, the local optima characterize global optima.

The solutions (4) and (5) and (6) coincide if $av_D = (n-1)v_A$.

Derivation of $p_D^i(x_i^*, y_i^*)$. The first-order conditions can be obtained from differentiating (9) after replacing $\Sigma_{i=1}^n ax_i$ by $\Sigma_{i=1}^n a_i x_i$ and (13) after replacing $\Sigma_{i=1}^n y_i$ by $\Sigma_{i=1}^n d_i y_i$ with respect to x_i and y_i as

$$\prod_{i=1}^n p_D^i \frac{1}{p_D^i}\frac{y_i}{(x_i+y_i)^2} q v_A = a_i \text{ for } i = 1, \ldots, n \tag{17}$$

$$\prod_{i=1}^n p_D^i \frac{1}{p_D^i}\frac{x_i}{(x_i+y_i)^2} v_D = d_i \text{ for } i = 1, \ldots, n. \tag{18}$$

For each i, (17) and (18) can be used to calculate $y_i = x_i \frac{a_i}{d_i}\frac{1}{q}$. Inserting this solution into the contest success function yields $p_D^i = \frac{a_i}{qd_i + a_i}$.

Notes

1. For this claim, see, for instance, Becker (1923/1960, 1) and Hartung (1930, 15). For a more general assessment, see also Andrews (1965). German generals, like Moltke the elder, had already been concerned about such a possibility in 1871 (see, e.g., Flammer 1966-1967, 207; Förster 1987).

2. Many other problems have similar features. The protection of dispersed natural resources such as fishing areas from illegal, or excessive, exploitation is one example. In this problem, the Coast Guard faces a formidable challenge in patrolling the vast fishing areas when resources are limited.

3. A local advantage of defense has, for instance, been claimed by Clausewitz (1832/1976, 84), who stated, "As we will show, defense is a stronger form of fighting than attack. [...] I am convinced that the

superiority of the defensive (if rightly understood) is very great, far greater than it appears at first sight." See also Bester and Konrad (2004, 1170) for further references that make this claim.

4. This assumption about perfect complementarity of battle victories for the defending player is chosen as an analytically convenient benchmark to study the importance of this complementarity. Moving away from this benchmark, the disadvantage of defending many fronts will weaken. Once the point of n independent contest with a prize (or a share in the aggregate prize) allocated in each of these battles is reached, the different contests become strategically independent games.

5. The value of winning is independent of the effort expended here. In some types of conflict, effort choices may reduce or increase the value of winning, as in Cooper and Restrepo (1967); Skaperdas (1992); and Baye, Kovenock and deVries (1998).

6. In the literature on conflict, equation (3) was first suggested by Tullock (1980), axiomatized by Skaperdas (1996), Kooreman and Schoonbeek (1997), and Clark and Riis (1998). Microfoundations of (3) are in Hirshleifer and Riley (1992, chap. 10) and Fullerton and McAfee (1999). This function, or variants of it, have been used extensively in the literature on conflict (see Garfinkel and Skaperdas [forthcoming] for an overview).

7. It should be mentioned that the attacking army often made the cost of losing the conflict for the inhabitants of the city that is under siege a function of their defense behavior and resistance, which complicates the picture further. Many instances reported, for example, by Runciman (1954/1994) for the period of the Crusades provide some illustrations.

8. According to Modelski (1964, 556), the concept of neighborhood as a prerequisite of "actual or potential opposition or enmity" has been acknowledged by early writers on the international system in the ancient Hindu world. Cusack and Stoll (1990) also acknowledge the principle of proximity for potential conflict in their conflict simulation studies.

9. A sufficient condition for the problem to reduce to a problem in simultaneous one-dimensional choices is that $a_i/d_i \equiv \gamma$ for all $i = 1, \ldots n$, with γ a positive finite constant.

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