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Multi-battle contests [☆]

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Abstract

We study equilibrium in a multistage race in which players compete in a sequence of simultaneous move component contests. Players may win a prize for winning each component contest, as well as a prize for winning the overall race. Each component contest is an all-pay auction with complete information. We characterize the unique subgame perfect equilibrium analytically and demonstrate that it exhibits endogenous uncertainty. Even a large lead by one player does not fully discourage the other player, and each feasible state is reached with positive probability in equilibrium (pervasiveness). Expected effort in the component contests may be non-monotonic in the closeness of the race and realized individual effort may exceed the value of the prize by a factor that is proportional to the maximum number of stage victories required.

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1. Introduction

In many contests final success or failure is determined not by the outcome of a single battle, but from multiple battles. For instance, in many contests players compete in a sequence of battles and the player whose accumulated battle victories first reach some predetermined minimum number (which may vary across players), is awarded the prize for victory in the overall contest. We call this structure a *multi-battle contest*.

Multi-battle contests have already received considerable attention in the area of R&D competition. Harris and Vickers (1987) described a patent race as a multi-battle contest: two players expend efforts on R&D in a sequence of single component contests. In each component contest one of the players wins, and the winner is determined as

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a stochastic function of the players' efforts in the respective component contest. The player who is first to win a given number of component contests wins the patent.¹

In politics, Klumpp and Polborn (2006) have applied multi-battle contests to explain the dynamics of candidate performance and campaign expenditures in the US: presidential primaries. In Klumpp and Polborn's model the winner of each state primary election is determined as a stochastic function of the players' campaign expenditures in that state. The player who is first to win a majority of the state elections wins the party's nomination. They use their findings to examine the parties' incentive to hold the state primary elections sequentially.

Sports is perhaps the most direct application. A tennis match, for instance, consists of a series of single battles. To be victorious, a player needs to win a certain number of sets before his or her competitor does. In the Major League Baseball World Series and the National Basketball Association's Finals playoff series two teams compete in a series of games with the winner being the first team to win four games. Similar examples exist in many other areas of sports competition.²

This article examines multi-battle contests in which the component contests are (first-price) all-pay auctions with complete information. An all-pay auction is a simultaneous move game in normal form in which players simultaneously submit bids (or efforts). Each player must pay his own bid (in contrast to winner-pay auctions) and the highest bidder wins the contest or prize (arbitrary tie-breaking rules are permitted). Examination of all-pay auctions dates back to at least Nalebuff and Stiglitz (1983),³ and the characterization of the one-stage all-pay auction and its applications has extensively been studied in recent years both for complete and incomplete information.⁴

The paper provides a complete characterization of the unique subgame perfect equilibrium of a wide class of symmetric and asymmetric two-player multi-battle contests with all-pay auction components. Players earn potentially different prizes for winning the overall contest and, in addition, identical non-negative intermediate prizes for winning each component contest. Empirically, the counterpart to these intermediate prizes in a patent race might be the benefit obtained from information spillovers or cost reduction in other production processes that arise from winning the component contest. In sports it might be the pride associated with winning individual games or sets. In political primary seasons it might be the prestige or bargaining power arising from controlling the delegates from a given state.

One implication of incorporating intermediate prizes is *pervasiveness*. With positive intermediate prizes, even a large lead by one player does not fully discourage the other player. Both players expend positive effort with positive probability in every component contest, no matter how far behind they might be. As a consequence, each player may win any given contest and each feasible state is reached with positive probability in equilibrium (*pervasiveness*). It is even possible for a lagging player to expend sufficient effort to catch up with his rival and possibly even become the leader again.⁵

We apply our general characterization results to illustrate two additional implications of multi-battle contests. First, individual and aggregate effort, as well as individual probabilities of winning, may be non-monotonic in the closeness of the race. In a battle that takes place in a state of full symmetry, where both players need the same number of component contest wins, players exert the most effort. They compete for both the intermediate prize and the final prize at this state. Both prizes are fully dissipated in expectation in this component contest, due to the cut-throat nature of this competition. If, instead, the multi-battle contest has moved away from this state of perfect symmetry,

¹ The multi-battle contests that we examine in this article are a special case of what Harris and Vickers (1985, 1987) call a *race*. According to Harris and Vickers (1987, p. 1) a race is "a competition in which a prize is awarded to the first competitor to achieve a given amount of progress." In Harris and Vickers (1985), players alternate in expending resources in order to move their position a certain distance, with the first player to succeed winning a prize. This is a game with complete and perfect information and does not fit our definition of a multi-battle contest. Other models of patent races that do not fit our definition of a multi-battle contest include Fudenberg et al. (1983), Leininger (1991), and Budd et al. (1993).

² For a survey of the theory of contests in sports see Szymanski (2003), where the fact that many different battles interact in sports is acknowledged.

³ Nalebuff and Stiglitz (1983, p. 41) cover this structure as a special case. Dasgupta (1986) applies a one-shot all-pay auction to model a patent race. In this article we argue that under certain conditions it is natural to consider dynamic games such as patent races, sports championship series, or sequential elections as multi-battle contests with component contests that are all-pay auctions.

⁴ These include Baye et al. (1996), Clark and Riis (2000), Ellingsen (1991), Hillman and Riley (1989) and Moldovanu and Sela (2001, 2006).

⁵ As suspense is one of the desirable features of sports events (see, e.g., Hoehn and Szymanski, 1999), this result may explain why such intermediate prizes are frequently observed in races which are carefully designed. In Formula 1 races, for instance, each Grand Prix generates some benefits to the winner, apart from the championship points that count for the overall championship that is awarded on an annual basis. Similarly, the PGA tour has large purses of prize money in the various tournaments, but each victory also contributes to the grand prize which is awarded at the end of the tour.

the advantaged player loses much from returning to the state of symmetry, due to the full dissipation occurring there, whereas the disadvantaged player does not gain much from returning to the state of symmetry for precisely the same reason. However, this asymmetry in the gain from winning causes effort to be non-monotonic when moving away from the state with perfect symmetry: with sufficient asymmetry of the state, players compete for the intermediate prizes only. Even if the advantaged player loses, he stays advantaged, and the process does not revert to a state of perfect symmetry. For this reason the players compete in an essentially symmetric battle for a prize that is equal to the intermediate prize. Close to the state of symmetry, the disadvantaged player competes for the intermediate prize only. Winning brings him back to the state of perfect symmetry, but this is not useful given the nature of the battle taking place there. The advantaged player has, in addition, to defend his advantage: For him the prize is equal to the intermediate prize plus his expected payoff from being advantaged. This makes the contest very asymmetric at such states and leads to even lower expected effort in the neighborhood of perfectly symmetric states.

Second, we show the potential for large overdissipation of rents in a multi-battle contest. A player always has the option of bidding 0. This rules out expected individual and aggregate overdissipation.⁶ However, multi-battle contests have equilibrium trajectories for which the individual effort can sum up to an amount that exceeds the value of the final prizes by an order of magnitude that is proportional to the number of component contests. Both the Harris–Vickers and Klumpp–Polborn multi-battle contest models also exhibit such overdissipation, although they do not address this issue explicitly or formally derive its magnitude. As noted by Baye et al. (1999), such probabilistic individual overdissipation also arises in other contests with complete information, such as the war of attrition or the sad loser auction. However, both of these contests have multiple equilibria, including asymmetric equilibria in which one player gives up immediately. In these equilibria, dissipation is obviously less than full, so the incidence of overdissipation requires an equilibrium selection argument.⁷ Our model avoids such an argument; probabilistic overdissipation arises in the unique equilibrium.

The all-pay auction captures the notion that random exogenous factors do not play a role in determining the outcome of a contest. Nonetheless, the outcomes are random due to the endogenous uncertainty generated by the use of nondegenerate mixed strategies in equilibrium. The degree of uncertainty is not exogenously determined, but is endogenously generated by the equilibrium strategies of the players. This constitutes a further difference compared to the frameworks in Harris and Vickers (1987) and Klumpp and Polborn (2006) who consider “lottery” component contests in which the probability that a player wins conditional on the vector of the two players’ efforts is equal to the ratio between the player’s expenditure and the sum of all contest expenditure in this component battle. The ‘lottery contest’ in Harris and Vickers (1987) is a special case of Lee and Wilde’s (1980) model of innovation with no discounting. Klumpp and Polborn also use the lottery contest for most of their analysis, but also report the results for cases with even more exogenous noise. Both Harris and Vickers (1987) and Klumpp and Polborn (2006) must resort to numerical methods to derive complete solutions for strategies and payoffs for all the multi-battle contests that they examine, except those involving a small number of component contests. In contrast, the all-pay auction framework provides closed form solutions for values and distributions at every stage, as well as transition probabilities between stages.

Contests with exogenous noise assume sufficient noise to generate nicely behaved payoff functions and pure strategy equilibria.⁸ For instance the Tullock (1980) contest equilibrium which has a probability function $p_A(a, b) = a^r / (a^r + b^r)$, where a and b are the respective efforts of players A and B , has been fully characterized only for r less than or equal to 2. For values of r greater than 2, characterization of equilibrium remains an open problem, although

⁶ We employ in the analysis that follows the asymmetric contest analogues of expected individual overdissipation (EIO) expected aggregate overdissipation (EAO), and probabilistic individual overdissipation (PIO) introduced by Baye et al. (1999). An equilibrium in our model exhibits EIO if an individual player’s expected expenditure exceeds his value of the prize. EAO arises when the expected sum of expenditures exceeds the highest value of the prize. PIO arises when there is a positive probability that an individual player’s expenditure exceeds his value of the prize.

⁷ Equilibrium refinements in the complete information war-of-attrition have been discussed by Kornhauser et al. (1989). See also Hörner and Sahuguet (2004) who obtain uniqueness by altering the war-of-attrition by having players alternate in expending effort in every other period and requiring that the rival match effort to avoid exit.

⁸ See also Malueg and Yates (2006) who generalize the contest success function in Klumpp and Polborn (2006) and derive results for a three-battle contest assuming the existence of a pure strategy equilibrium.

it is clear that equilibrium requires nondegenerate mixed strategies (see, e.g., Baye et al., 1994).⁹ The all-pay auction is the case of $r = \infty$, which is the only value of $r > 2$ for which a complete characterization of the set of Nash equilibria exists. The same applies for the Lazear–Rosen (1981) difference form model with small noise. In this framework, Che and Gale (2000) provide a partial characterization of the one-shot Nash equilibrium for a specific type of exogenous uncertainty. A full characterization of equilibrium for contests with small noise remains, up to now, an open question. This makes the case with no noise a particularly relevant benchmark case.

Even if one could avoid the analytical difficulties inherent in use of exogenous noise models as component games, the all-pay auction can be useful as a benchmark contest success function. The all-pay auction is the counterpart to homogeneous product Bertrand price competition with constant unit cost in the context of competition in sunk expenditures (like effort, devotion of time, or use of scarce resources). Classic Bertrand competition has had widespread application as a model of “cutthroat competition” because of its property that it only takes two equally efficient firms to dissipate all rent (profit). The same is true of the all-pay auction as a game of sunk expenditure.

2. A multi-battle contest

Consider two players A and B . The players take part in a race which is comprised of a sequence of one shot simultaneous move component contests (“battles”). The series of component contests awards a prize to the winner of the race, and this prize is valued at Z_A and Z_B by these players. We assume that $Z_A \geq Z_B > 0$. In order to win the prize, player A must win n of these component contests before player B wins m component contests, with $(n, m) \gg 0$. Similarly, B receives the prize Z_B for winning the race if he wins m component contests before A wins n contests. In addition to the final prize Z_A or Z_B , intermediate prizes are awarded to the winner of each component contest. We assume here that each of these intermediate prizes is valued at $\Delta \geq 0$ by both contestants.

Starting from the initial state (n, m) the first contest, C^{nm} , is played. If player A wins the contest, he receives the intermediate prize $\Delta \geq 0$ and the state moves to $(n - 1, m)$, indicating that in order to win the race player A then only needs to win $n - 1$ component contests before B wins m component contests. If player B wins the first contest, B receives the intermediate prize Δ and the state moves to $(n, m - 1)$, indicating that player B only needs to win $m - 1$ more contests in order to win the race. The outcome of each component contest becomes public information at its conclusion. In each state, the players simultaneously decide upon their expenditures in the next contest. The solution concept that we will use is subgame perfect equilibrium. In each state (i, j) we shall denote by $v_A(i, j)$ and $v_B(i, j)$ the subgame perfect equilibrium continuation values of players A and B , respectively, which is the value the player attributes to starting the race at this state. We argue below that these continuation values are well-defined.

Since a player wins the race if he is the first to reach a position with no component contests left to win, we may set

$$\begin{aligned} v_A(0, j) &= Z_A \text{ and } v_B(0, j) = 0 \quad \text{for all } j > 0, \\ v_B(i, 0) &= Z_B \text{ and } v_A(i, 0) = 0 \quad \text{for all } i > 0. \end{aligned} \tag{1}$$

At any (i, j) that is not such an end state, a component contest takes place. Each of the component contests, C^{ij} , is described as follows. Players simultaneously choose efforts, denoted $a \geq 0$ and $b \geq 0$, respectively. A player who expends a strictly higher effort than his opponent wins the component contest. If the players expend the same effort, for simplicity we assume that player A wins the contest at (i, j) if $v_A(i - 1, j) > v_B(i, j - 1)$ and player B wins the contest if the inequality is reversed. In the event players expend the same effort and $v_A(i - 1, j) = v_B(i, j - 1)$ we assume that the winner of the component contest is chosen by a fair randomizing device.¹⁰

Fig. 1 illustrates a race for $n = 6$ and $m = 4$. Players start at $(6, 4)$, can reach any of the dark states (i, j) with $i \leq 6$ and $j \leq 4$, and will finally end on the upper or right boundary. One of the issues we address is whether the race is *pervasive* in the sense that, starting from any (i', j') , all states (i, j) with $i \leq i'$ and $j \leq j'$ are reached with positive probability in the subgame perfect equilibrium. This is in contrast to some models of patent races where a sufficient lead by one contestant leads the other contestant to give up.

⁹ To assess what constitutes “sufficient noise” in the Tullock contest, note that when player A expends effort a and player B expends $b = \theta a$ with $\theta < 1$, the probability that A loses is $[1 + \theta^{-r}]^{-1}$. Hence if B exerts 1/2 the effort of A , player A loses with probability $[1 + 2^r]^{-1}$. For $r = 2$ this is 1/5, and for $r \leq 1$, the cases examined by Klumpp and Polborn (2006), this is at least 1/3.

¹⁰ This tie breaking rule only affects equilibrium behavior when $\Delta = 0$. See Section 3 for a discussion of alternative tie-breaking rules.

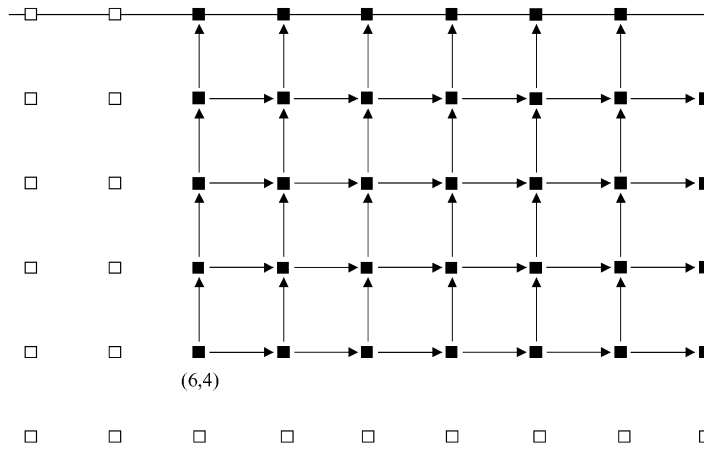


Fig. 1. A race with $(n, m) = (6, 4)$.

Starting or continuing from the state $(n, m) = (1, 1)$, which we will term the “decisive state,” this contest is just a standard all-pay auction with complete information with prizes $z_A(1, 1) \equiv Z_A + \Delta$ and $z_B(1, 1) \equiv Z_B + \Delta$. It is now well known (see Hillman and Riley, 1989 and Baye et al., 1996) that the two player all-pay auction has a unique equilibrium in mixed strategies. The following proposition characterizes the unique equilibrium in the all-pay auction between two contestants with these valuations.

Proposition 1. (See Hillman and Riley, 1989.) Let $\Gamma(z_A, z_B)$ be a two-player first price all-pay auction with prize values z_A and z_B , where $z_A \geq z_B > 0$. $\Gamma(z_A, z_B)$ has a unique Nash equilibrium in mixed strategies. In this equilibrium players’ efforts (bids) are chosen randomly according to the cumulative distribution functions

$$F_A(a) = \begin{cases} \frac{a}{z_B} & \text{for } a \in [0, z_B], \\ 1 & \text{for } a > z_B \end{cases} \tag{2}$$

and

$$G_B(b) = \begin{cases} \frac{z_A - z_B}{z_A} + \frac{b}{z_A} & \text{for } b \in [0, z_B], \\ 1 & \text{for } b > z_B. \end{cases} \tag{3}$$

Equilibrium payoffs are $u_A^* = z_A - z_B$ and $u_B^* = 0$. In equilibrium, the expected sum of the efforts is $E(a + b) = \frac{1}{2}z_B[1 + \frac{z_B}{z_A}]$ and the probability of winning the prize is $p_A = 1 - \frac{z_B}{2z_A}$ and $p_B = \frac{z_B}{2z_A}$, for players A and B respectively.

Before proceeding we should note a subtlety concerning the issue of tie-breaking rules. Proposition 1 assumes that both players’ prize values are strictly positive. Under this assumption, since both players employ nondegenerate mixed strategies in the unique equilibrium, the analysis of the Proposition is invariant with respect to the choice of tie-breaking rule applied in the event that players expend the same effort. Hence, there is no loss of generality in specifying a particular tie-breaking rule. Similarly, if both players assign a zero value to the prize ($z_A = z_B = 0$), the unique equilibrium has both players setting their bid equal to 0 with probability 1, regardless of the tie-breaking rule employed. Indeed, the choice of a strictly positive bid in this case is strictly dominated. Finally, if one player has a positive value of the prize while the other has a 0 value, say $z_A > z_B = 0$, then the choice of the tie-breaking rule is crucial. In any equilibrium player B must expend zero effort, since a positive effort is strictly dominated. However, any tie-breaking rule that does not allocate the prize with certainty to player A at $(0, 0)$ will lead to the non-existence of equilibrium, in pure or mixed strategies. In this case player A would like to outbid player B by the smallest possible amount, but there is no smallest possible number greater than 0. Allocating the prize to player A with certainty avoids this open endedness problem.¹¹

¹¹ If $z_A > z_B = 0$ and we assume a sufficiently fine but finite grid of potential effort levels with mesh ϵ that includes 0, $a = \epsilon > b = 0$ is the unique Nash equilibrium (unless in the event of a tie at $(0, 0)$ player A wins with sufficient probability p that $\epsilon/z_A \geq (1 - p)$). Moreover, player A would

According to Proposition 1, the player who attributes a higher value to winning has a payoff equal to the difference between his own and his rival's value of winning, and the other player has an equilibrium payoff that is equal to zero. This determines the values that the players attribute to being at the state (1, 1) as

$$\begin{aligned} v_A(1, 1) &= \max[z_A(1, 1) - z_B(1, 1), 0] \\ &= [(v_A(0, 1) - v_A(1, 0)) + \Delta] - [(v_B(1, 0) - v_B(0, 1)) + \Delta] \\ &= Z_A - Z_B, \end{aligned} \tag{4}$$

and similarly,

$$v_B(1, 1) = 0. \tag{5}$$

Note that the size of the intermediate prize Δ does not affect the continuation values $v_A(1, 1)$ and $v_B(1, 1)$. It does, however, affect the equilibrium distribution of efforts and the respective probabilities of winning the contest C^{11} . From Proposition 1 the expected sum of the efforts in C^{11} is $\frac{1}{2}(Z_B + \Delta)[1 + \frac{Z_B + \Delta}{Z_A + \Delta}]$ and the respective probabilities of winning are $p_A = 1 - \frac{Z_B + \Delta}{2(Z_A + \Delta)}$ and $p_B = \frac{Z_B + \Delta}{2(Z_A + \Delta)}$.

Define $\Sigma(k) = \{(i, j) \gg 0: i + j = k\}$. Now that the continuation values at (1, 1), (2, 0) and (0, 2) are uniquely defined, we can consider the states in $\Sigma(3)$, (2, 1) and (1, 2). The contest C^{21} can either lead to (1, 1) or to (2, 0), and the value of winning this component contest is equal to the intermediate prize Δ plus the absolute value of the difference in the respective contestant's continuation values at (1, 1) and (2, 0), which are uniquely determined. Hence, using subgame perfection, C^{21} reduces to a problem that is equivalent to a standard all-pay auction with complete information, which again has a unique equilibrium which is determined analogously to the equilibrium for C^{11} , and which uniquely determines the continuation values $v_A(2, 1)$ and $v_B(2, 1)$. Similar reasoning applies for C^{12} , and the continuation values for the end states (0, 3) and (3, 0) are also well defined by (1). More generally, in any component contest C^{ij} the value of the "prize" of winning the contest is equal to Δ plus the absolute value of the difference in the respective player's continuation values in states $(i - 1, j)$ and $(i, j - 1)$. Formally, in C^{ij} players play an all-pay auction with prizes

$$\begin{aligned} z_A(i, j) &= v_A(i - 1, j) - v_A(i, j - 1) + \Delta \quad \text{and} \\ z_B(i, j) &= v_B(i, j - 1) - v_B(i - 1, j) + \Delta. \end{aligned} \tag{6}$$

This illustrates how unique continuation values for all states (n, m) can be calculated recursively.

More generally, in order to characterize the nature of the subgame perfect equilibrium in the multi-battle contest we first define the set of "separating states."

Definition 1. Suppose $Z_A \geq Z_B > 0$ and consider the set of states $(i, k - i) \in \Sigma(k)$ for $k \geq 2$. We define a state $(i_k, k - i_k)$ to be a separating state if it has the following separating property:

$$\begin{aligned} v_A(i, k - i) &= Z_A \text{ and } v_B(i, k - i) = 0 \quad \text{for all } i < i_k, \\ v_A(i, k - i) &= 0 \text{ and } v_B(i, k - i) = Z_B \quad \text{for all } i > i_k. \end{aligned} \tag{7}$$

The separating property provides some structure to the state space and has implications for the continuation values of the separating states themselves. One implication is that there can be at most two separating states for each $\Sigma(k)$. This follows by contradiction: suppose $(i_k, k - i_k)$, $(j_k, k - j_k)$ and $(l_k, k - l_k)$ are separating states in $\Sigma(k)$, and let $i_k > j_k > l_k$. Then, by the separating property of i_k , $v_A(j_k, k - j_k) = Z_A$ and by the separating property of l_k , $v_A(j_k, k - j_k) = 0$; hence, a contradiction. When there are two separating states (i, j) and (i', j') with $i > i'$ in $\Sigma(k)$, they must be neighboring in the sense that $i = i' + 1$, and the continuation values at these states need to be $v_A(i, j) = 0$, $v_B(i, j) = Z_B$, $v_A(i', j') = Z_A$, $v_B(i', j') = 0$. This structure and the neighboring property are useful in proving the main proposition of the paper.

win the prize with certainty at the smallest positive bid. As ϵ goes to 0, this equilibrium approaches the equilibrium of the continuous strategy space game with the tie-breaking rule that we have chosen.

Proposition 2.

- (i) For every $k \geq 2$ there exist one or two separating states in $\Sigma(k)$.
- (ii) The state $(i, j) \gg 0$ is a separating state if and only if $\frac{j-1}{i} \leq \frac{Z_B}{Z_A} \leq \frac{j}{i-1}$. Hence, if there exists an $(i, j) \in \Sigma(k)$ such that $\frac{j-1}{i} < \frac{Z_B}{Z_A} < \frac{j}{i-1}$ then (i, j) is the unique separating state in $\Sigma(k)$.
- (iii) $(i, j + 1)$ and $(i + 1, j)$ comprise the set of separating states in $\Sigma(k)$ if and only if $\frac{Z_B}{Z_A} = \frac{j}{i}$ for $(i, j) \in \Sigma(k - 1)$.
- (iv) For any (i, j) , $v_A(i, j) = \min(Z_A, \max(0, jZ_A - iZ_B))$ and $v_B(i, j) = \min(Z_B, \max(0, iZ_B - jZ_A))$. If (i_k, j_k) is a separating state then $v_A(i_k, j_k) = \max(0, j_kZ_A - i_kZ_B) \leq Z_A$ and $v_B(i_k, j_k) = \max(0, i_kZ_B - j_kZ_A) \leq Z_B$.
- (v) An immediate consequence is that $v_A(i, j) > 0$ if and only if $\frac{Z_B}{Z_A} < \frac{j}{i}$ and $v_B(i, j) > 0$ if and only if $\frac{Z_B}{Z_A} > \frac{j}{i}$.

The proof of Proposition 2 is relegated to Appendix A. Fig. 2 illustrates the structure of the problem, with $(i_k, k - i_k)$ a unique separating state in $\Sigma(k)$.

Most non-separating states in the set $\Sigma(k + 1)$ simply inherit their continuation values from the fact that the component contest in this state leads to two possible states which do not differ in the continuation values for players. Accordingly, the component contest in such a state is symmetric, and is essentially about the intermediate prize Δ only. The situation for separating states and for states next to separating states is more complex. If $(i_k, k - i_k)$ is a unique separating state in $\Sigma(k)$, then the separating state in $\Sigma(k + 1)$ is either the state to the left of $(i_k, k - i_k)$ or the state right below $(i_k, k - i_k)$, or both. The latter happens in the non-generic case in which the separating state $(i_k, k - i_k)$ is located right on the line through $(0, 0)$ with slope Z_B/Z_A . Since on this line $v_A(i, j) = v_B(i, j) = 0$, we call this line the *complete dissipation line*. If $\Sigma(k)$ has two separating states, then the state in $\Sigma(k + 1)$ that is located next to these two states is the (unique) separating state in $\Sigma(k + 1)$.

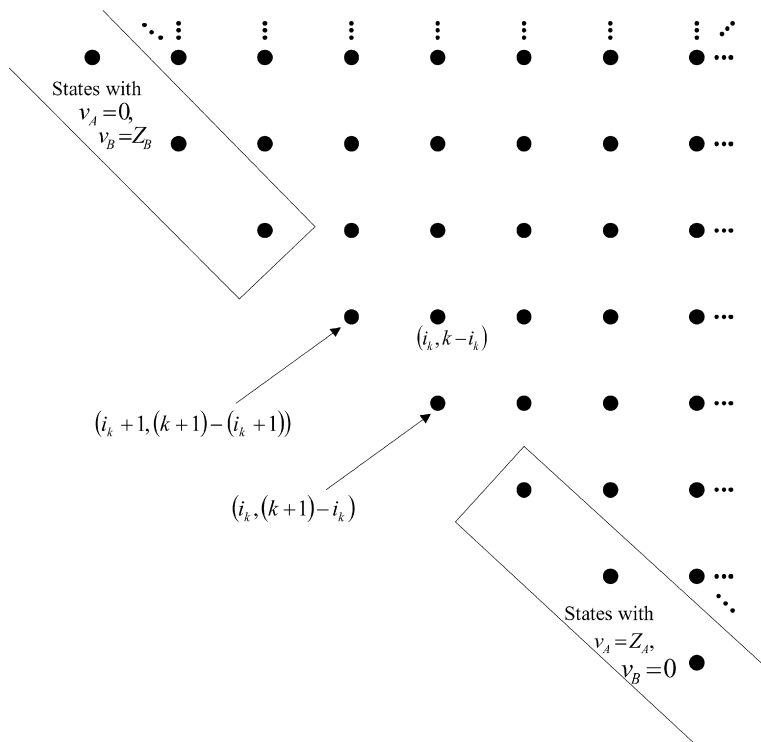


Fig. 2. Potential separating states in $\Sigma(k + 1)$.

To facilitate our analysis of the equilibrium distributions define a *non-trivial component contest* as a component contest in which both contestants expend positive effort with positive probability, and a *trivial component contest* as a contest in which both contestants expend zero contest effort with probability one.¹²

3. No intermediate prizes ($\Delta = 0$)

The special case with terminal prizes, Z_A and Z_B , but no intermediate prizes, $\Delta = 0$, is of particular interest. All of the results in Proposition 2 hold for this special case. Indeed, for $\Delta = 0$ the equilibrium distributions take a very simple form. The following corollary summarizes the characterization of these distributions in this special case:

Corollary 1. *Suppose $Z_A \geq Z_B > 0$ and $\Delta = 0$. A non-trivial component contest occurs at (i, j) if and only if $v_A(i - 1, j) > 0$ and $v_B(i, j - 1) > 0$, which holds if and only if $\frac{i-1}{i} < \frac{Z_B}{Z_A} < \frac{j}{i-1}$. If $\frac{Z_B}{Z_A} \leq \frac{i-1}{i}$ then, starting in state (i, j) , player A is able to win the remaining contests with no effort. If $\frac{Z_B}{Z_A} \geq \frac{j}{i-1}$ then starting in (i, j) , player B is able to win the remaining contests with no effort.*

Note that one implication of the corollary is the following: Since for a given k the sets $\{[\frac{k-i-1}{i}, \frac{k-i}{i-1}), i = 1, \dots, k - 1\}$ partition the interval $[0, \infty)$ and for each of the states $(i, j) \in \Sigma(k)$, $\frac{j}{i}$ lies in exactly one interval, at most one such state generates a non-trivial contest. That is, the inequalities $\frac{i-1}{i} < \frac{Z_B}{Z_A} < \frac{j}{i-1}$ can jointly hold for at most one such state. However, it is possible that the inequalities do not jointly hold for any such state. This occurs when there are two separating states in $\Sigma(k)$. That is, $\frac{Z_B}{Z_A}$ takes the value of one of the non-zero endpoints of these intervals: $\frac{Z_B}{Z_A} = \frac{k-i-1}{i}$, for some $i \in 1, \dots, k - 2$. In this case, for $i' \leq i$, player A wins at $(i', k - i')$ without having to expend any further effort and for $i' > i$, player B wins at $(i', k - i')$ without having to expend any further effort. Both $(i, k - i)$ and $(i + 1, k - 1 - i)$ are separating states in $\Sigma(k)$. Note that this guarantees that the state $(i + 1, k - i)$ is a separating state in $\Sigma(k + 1)$ at which there is a non-trivial contest.

Fig. 3 shows the separating states for the set of initial states $(i, j) \in \Sigma(k)$, $k \leq 19$, and with prize values $Z_A = 11$, $Z_B = 5$, and $\Delta = 0$. The states indicated with black boxes are those with non-trivial component contests. The valuations of winning at these states and the resulting equilibrium play that follows from these are characterized in Appendix B. The states labeled A and B are separating states in $\Sigma(17)$, but each have trivial component contests. At state B, $z_B > 0$, $z_A = 0$, and player B wins this and all remaining contests with no effort. At point A, $z_A > 0$, $z_B = 0$, and player A wins all remaining contests with no effort. As noted above, existence of equilibrium at A or B relies on the tie-breaking rule, as do the equilibrium trajectories inside the areas of states indicated with white boxes.

Before moving on to examine several of the general properties of the race when $\Delta > 0$ a few more remarks are in order for the case where $\Delta = 0$. First, the treatment of asymmetric per unit costs of effort when $\Delta = 0$ is especially straightforward. Since behavior is invariant with respect to positive affine transformations of utility, we may incorporate asymmetric constant per unit effort costs in our model by dividing each contestant's utility by the corresponding contestant's per unit effort cost. Hence, if the prize values for A and B are Z_A and Z_B , respectively, and the corresponding per unit costs of effort are c_A and c_B , then the equilibrium of the game parameterized by (Z_A, Z_B, c_A, c_B) is identical to that of a game with unit cost equal to one for both players and transformed values $\tilde{Z}_A = \frac{Z_A}{c_A}$ and $\tilde{Z}_B = \frac{Z_B}{c_B}$. With $\Delta = 0$, all of our previous results then go through with these values inserted in place of Z_A and Z_B .¹³ Although the treatment of asymmetric unit costs of effort can be carried out through backward induction in the case where $\Delta > 0$, the analysis requires somewhat more involved calculations, since transforming a contestant's utility by dividing by the unit cost of effort not only changes the terminal prizes, but also changes each component contest prize. If $c_A \neq c_B$, then $\frac{\Delta}{c_A} \neq \frac{\Delta}{c_B}$, and the component contest prizes become asymmetric in the transformed game. This means that the component contest prize values do not net out in calculating continuation values, so that the continuation value at any (i, j) is a complicated function of component contest prize values at states $(i', j') \leq (i, j)$ as well as the terminal prizes.

¹² It is straightforward to show that, given the continuous strategy space and the tie-breaking rule that we employ, it cannot be equilibrium behavior for one contestant to expend positive effort with positive probability and the other to expend positive effort with probability zero.

¹³ However, if $\tilde{Z}_B > \tilde{Z}_A$ the indices in the analysis will have to be reversed, since the analysis has assumed that $Z_A \geq Z_B$.

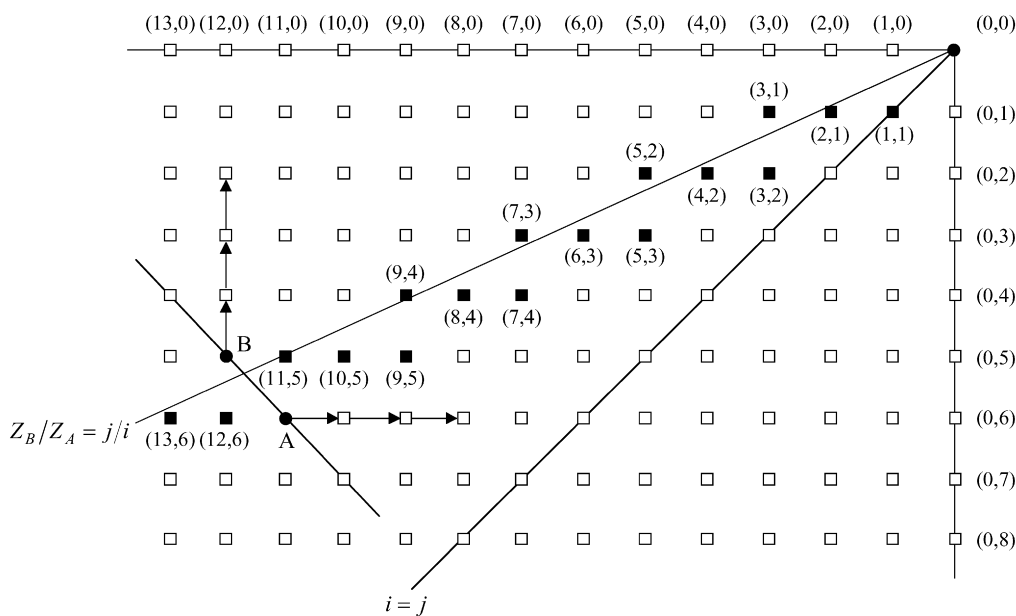


Fig. 3. Separating states in $\Sigma(k)$, $k \leq 19$, for $Z_B/Z_A = 5/11$ and $\Delta = 0$.

The game with $\Delta = 0$ is also useful in illustrating the potential for overdissipation of rents in a multi-battle contest. Since only one player wins the terminal prize, the maximum possible rent to be earned in this game is Z_A . Since a player has the right to opt out of the contest by bidding zero at every nonterminal state, for any such multi-battle contest (m, n) , in equilibrium there can be no player whose expected effort exceeds the player's value of the prize. Moreover, the expected sum of efforts cannot exceed Z_A .¹⁴ However, because of the dynamic nature of the model, unlike the one shot first price all-pay auction, there are many parameter specifications for which, in equilibrium, there is a positive probability that an individual contestant will expend a higher cost of effort than the contestant's value of the prize.¹⁵ To illustrate this, suppose that $Z_B = \phi Z_A$ where $1 > \phi > \frac{n-1}{n}$, and look at the contest that starts at the initial state (n, n) . The "decisive state" $(1, 1)$ is reached if and only if player B wins every contest starting from the states (m, m) , $n \geq m \geq 2$, and player A wins every contest starting from a state $(m, m - 1)$, $n \geq m \geq 2$. (See Fig. 4.) This in some sense represents a situation of maximal dissipation in the game starting at (n, n) , because the contest remains non-trivial for the longest possible time. We show in Appendix C that equilibrium trajectories can be found in which the total effort expended by a single player sums up to an amount with a supremum of nZ_A as $\phi \rightarrow 1$. Intuitively, the process may follow the path along the states with non-trivial contests as in Fig. 4 with positive probability, and, if the players are very similar, each player may expend close to 0 in contests $C^{m, m-1}$ and close to $Z_B \approx Z_A$ in contests $C^{m, m}$.

Hence, the least upper bound on the degree of possible individual overdissipation in a realization of the subgame perfect equilibrium strategies can be quite large. Note that this cannot arise in either the one-shot all-pay auction or the version of our model with $Z_A = Z_B = 0$ and $\Delta > 0$. The existence of at least one prize of positive value which is captured as a result of a sequence of expenditures is crucial to the result. Overdissipation arises because "sunk costs are sunk costs."¹⁶ Expenditures arising in the past have no effect on a contestant's willingness to expend effort to capture the terminal prize from a given state (i, j) . No matter what a contestant's past expenditure, contestant A is

¹⁴ These definitions of overdissipation are the asymmetric contest analogues of Expected Individual Overdissipation (EIO) and Expected Aggregate Overdissipation (EAO) introduced by Baye et al. (1999). A symmetric equilibrium exhibits Expected Individual Overdissipation if an individual player's expected bid exceeds the value of the prize. Expected Aggregate Overdissipation arises when the expected sum of payments of the players exceeds the value of the prize.

¹⁵ In the context of symmetric contests, Baye et al. (1999) refer to this as Probabilistic Individual Overdissipation (PIO).

¹⁶ This result is not unique to the all-pay auction structure. Extreme overdissipation along some equilibrium trajectories can emerge in the multi-battle games analyzed by Harris and Vickers (1987) and Klumpp and Polborn (2006), but has not been emphasized in this work.

An implication of these considerations is that the equilibrium correspondence is generally not upper-hemicontinuous at $\Delta = 0$ under tie-breaking rules that do not allocate the component contest win to a player with a strictly higher prize in the component contest. Among tie-breaking rules that allocate the win to a player with a strictly higher prize value, equilibrium strategies, expected total effort and payoffs are continuous at $\Delta = 0$, although the set of equilibrium probability distributions over paths of contest play will generally violate lower-hemicontinuity. As $\Delta \rightarrow 0$, component contests (i, j) for which $z_A(i, j) = z_B(i, j) = \Delta$ involve nondegenerate mixed strategies under which each player has probability $\frac{1}{2}$ of winning. In the limit, when $z_A(i, j) = z_B(i, j) = \Delta = 0$, equilibrium is in pure strategies with $(a, b) = (0, 0)$, and, depending on the tie-breaking rule, the probability of winning may vary between 0 and 1. The case where this probability is $\frac{1}{2}$, say due to ties being broken with a fair coin flip, generates a probability distribution over paths that is the limit of the distributions as $\Delta \rightarrow 0$.

4. Pervasiveness and the non-monotonicity of effort with $\Delta > 0$

Note that $\Delta > 0$ implies that each contestant has a positive value of winning at any state $(i, k - i) \gg 0$. An immediate consequence is then

Corollary 2. *If $\Delta > 0$ a non-trivial contest occurs at all points $(i, k - i) \gg 0$.*

Corollary 2 reveals that the intermediate prizes are important to obtain a positive contest effort if players are in states that are some distance from the separating states. Intermediate prizes avoid contests becoming trivial. Consider sports contests. Intermediate prizes may consist of purely psychological rewards or ego-rents. For instance, a player or team who already leads by a large margin may enjoy a further increase in his lead, making his victory even more spectacular, or a player who is close to final defeat may enjoy some reward from winning at least another single battle, showing that he or she is at least a serious competitor. Moreover, in many sports contests, monetary prizes are attached to battle victories. The winner of a single Grand Prix Formula I race receives at least a cup and some reward in terms of increased market value and sponsoring contracts, and in tennis or golf tournaments considerable prize money is at stake in each single tournament. A comparison of Corollaries 1 and 2 shows that such intermediate prizes are important to avoid the series of battles becoming rather uninteresting once one of the players has accumulated a sufficient advantage that the other player gives up.

An interesting question in races is whether the current state of the race uniquely determines how the race evolves. Particularly in the literature on patent races, the point has been made that a lead by one contestant can be sufficient to guarantee that this contestant also wins the final prize with probability 1. Intuitively, if contestant A leads by sufficiently many component contest wins, then B gives up, knowing that any effort B might make to catch up with A will be rendered useless if A may react by increasing his effort to keep B at a distance all the way to the finish line. This is not the case with $\Delta > 0$. Note that B has a strictly positive probability of winning for any $(i, k - i) \gg 0$. More generally speaking, we define the race as *pervasive* if the equilibrium probability that state $(i', k' - i')$ is reached starting from a given $(i, k - i) \geq (i', k' - i')$ is strictly positive. We conclude from Proposition 1:

Corollary 3. *The multi-battle contest with $\Delta > 0$ is pervasive.*

Indeed, it is possible to characterize completely the nature of the equilibrium local strategies employed in any particular component contest (i, j) . The following corollary determines the nature of these distributions in states (i, j) which cannot lead to a separating state after a single component contest.

Corollary 4. *The transition probability from an interior state $(i, k - i) \gg 0$ to $(i, k - i - 1)$ and to $(i - 1, k - i)$ is equal to $1/2$ for all $(i, k - i)$ and $\Delta > 0$ for which $i_{k-1} \notin \{i, i - 1\}$.*

and consequently $v_B(i, j - 1) = 0$. Since $v_A(i, 0) = 0$, there must be some maximum $j' < j - 1$ such that $v_A(i, j') < Z_A$. This implies that at $(i, j' + 1)$, $z_A(i, j' + 1) > 0$, and $z_B(i, j' + 1) = 0$. Hence, at $(i, j' + 1)$, both players bid 0 and player A must win with certainty, moving the state to $(i - 1, j' + 1)$. At $(i - 1, j' + 1)$, either $z_A(i - 1, j' + 1) > 0$, and $z_B(i - 1, j' + 1) = 0$, whereupon A wins again with certainty with $(a, b) = (0, 0)$, or both players have zero prizes, in which case we repeat the same argument. Since there are a finite number of states, eventually this iteration leads to a terminal state $(0, j'')$ where $v_A = Z_A$.

Proof. The separating property (7) of i_{k-1} implies that $v_A(i, k - i - 1) = v_A(i - 1, k - i)$ and $v_B(i, k - i - 1) = v_B(i - 1, k - i)$ if $i_{k-1} \notin \{i, i - 1\}$. Accordingly, $z_A = z_B = \Delta$ at $(i, k - i)$, and this implies that the equilibrium of the component contest is symmetric at $(i, k - i)$. \square

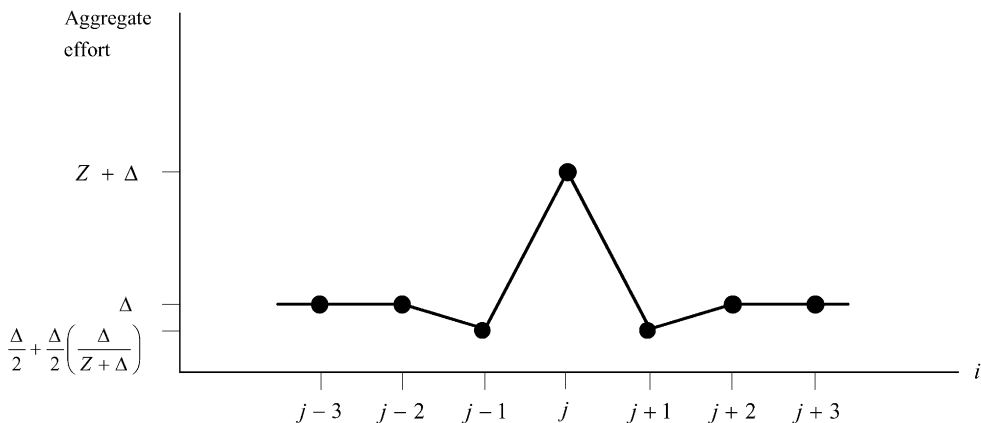
Corollary 4 suggests that a contestant who is lagging far behind for some time and is only one or two battles away from final defeat may still move back towards the range of separating states, and may even win the final prize, and this probability is increasing in the size of the intermediate prize. Hence, intermediate prizes are rather important for producing suspense.

A complete characterization of equilibrium local strategies employed in separating states and states that are within one component contest outcome of a separating state is easily obtained by inserting the values for $v_A(i, j)$ and $v_B(i, j)$ derived in Proposition 2 into the expression for state (i, j) prizes in equation (6), and then applying Proposition 1. For illustrative purposes, an important special case of our analysis is that in which $Z_A = Z_B$ and $\Delta > 0$, which can be found in Appendix D. With $Z_A = Z_B \equiv Z$ and $\Delta > 0$, across component contests the expected individual and aggregate effort of the two contestants and the individual contestant win probabilities are non-monotonic in the ratio $\frac{j}{i}$. Panel A of Fig. 5 shows the expected aggregate effort as a function of i for a given j , assuming that all of the relevant points are still interior. Intuitively, at a symmetric state $i = j$ the prize that is at stake is maximal and equal to $Z + \Delta$, and is fully dissipated in the equilibrium, due to symmetry. At $i = j \pm k$ for $k > 1$ the players essentially compete for Δ in, what is essentially a symmetric contest for this prize. There is one important difference at states $i = j - 1$ and $i = j + 1$. Here, the advantaged player has much to lose from returning to a completely symmetric state, whereas the prize for the disadvantaged player still remains as low as Δ . This makes the contest very asymmetric at this state and causes aggregate effort to be smaller than Δ in expectation.

Panel B of Fig. 5 shows how this distribution of prizes translates into equilibrium win probabilities for A. Player A wins the contest with probability $\frac{1}{2}$ for any interior state $i \leq j - 2$, but his win probability increases to $1 - \frac{\Delta}{2(Z+\Delta)}$ at $i = j - 1$, decreases again to $\frac{1}{2}$ at $i = j$, and further decreases to $\frac{\Delta}{2(Z+\Delta)}$ at $i = j + 1$, before increasing and remaining at $\frac{1}{2}$ for all interior states with $i \geq j + 2$.

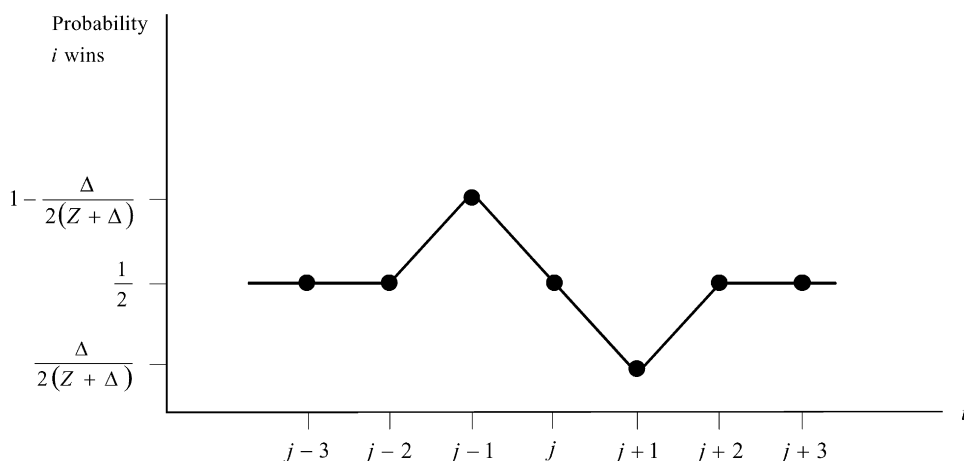
5. Conclusions

We have provided a complete analytical solution for the unique subgame perfect equilibrium of a type of race in which players interact repeatedly in (simultaneous move) all-pay auctions. We found that there are states in such a race that have a separating property: there exists at least one player who obtains a considerable rent from winning the battle due to his resulting advantaged position in the overall race. Despite this, as long as the intermediate prize values are positive, conflict does not completely vanish outside of these separating states, even though a long series of battle victories may be required for either of the players to reach final victory from this state. The final winner of the overall contest is not readily determined as an outcome of the battle in a separating state. The player who is lagging



Panel A: Expected aggregate effort at (i, j) as a function of i for given j assuming that all relevant points are interior.

Fig. 5. Non-monotonicity of effort and win probabilities ($Z_A = Z_B \equiv Z, \Delta > 0$).



Panel B: The probability that contestant i wins the component contest at (i, j) .

Fig. 5. (continued)

behind may catch up, and does catch up with a considerable probability in the equilibrium, depending on the value of the intermediate prizes that are allocated to the winners of the component contests. In this sense, intermediate prizes make the race pervasive: given any initial state of the race (i, j) , any state (i', j') with $(i, j) \geq (i', j')$ is reached with strictly positive probability. The existence of intermediate prizes also causes expected individual and aggregate component contest effort and individual win probabilities to be nonmonotonic in the magnitude of a player's lead in the race. Even without intermediate prizes, with strictly positive probability the game moves along trajectories of battle victories over which each player's realized effort far exceeds the value that is at stake.

Appendix A. Proof of Proposition 2

We first prove by induction that for each $k = 1, 2, 3, \dots$ there exists at least one $i_k \in \{1, \dots, k - 1\}$ that has the separating property described in Definition 1 and possesses the property that

$$\begin{aligned} (k - i_k)Z_A - i_k Z_B &\leq Z_A, \\ i_k Z_B - (k - i_k)Z_A &\leq Z_B. \end{aligned} \tag{8}$$

We also demonstrate that these states have continuation values

$$\begin{aligned} v_A(i_k, k - i_k) &= \max(0, (k - i_k)Z_A - i_k Z_B), \\ v_B(i_k, k - i_k) &= \max(0, i_k Z_B - (k - i_k)Z_A). \end{aligned} \tag{9}$$

The properties of these states are then used to demonstrate claims (i) through (v) of Proposition 2.

Note that the property holds for $k = 2$: $v_A(0, 2) = Z_A$, $v_B(0, 2) = 0$, $v_A(2, 0) = 0$ and $v_B(2, 0) = Z_B$. Moreover, by Proposition 1, $v_A(1, 1) = Z_A - Z_B = \max(0, 1 \cdot Z_A - 1 \cdot Z_B)$, with $1 \cdot Z_A - 1 \cdot Z_B \leq Z_A$, and $v_B(1, 1) = 0 = \max(0, Z_B - Z_A)$, and $Z_B - Z_A \leq Z_B$.

Assume now that a separating state $(i_k, k - i_k)$ exists in $\Sigma(k)$ with $(k - i_k)Z_A - i_k Z_B \leq Z_A$ and $i_k Z_B - (k - i_k)Z_A \leq Z_B$ and continuation values as in (9). Let this separating state be depicted in Fig. 2. Turn to states $(i, j) \in \Sigma(k + 1)$, which are the points at the south-west frontier of the set of points in Fig. 2. We show that, then, a separating state $(i_{k+1}, k + 1 - i_{k+1})$ exists such that this state has the separating property (7) and fulfills

$$\begin{aligned} ((k + 1) - i_{k+1})Z_A - i_{k+1}Z_B &\leq Z_A, \\ i_{k+1}Z_B - ((k + 1) - i_{k+1})Z_A &\leq Z_B \end{aligned} \tag{10}$$

and

$$v_A(i_{k+1}, (k+1) - i_{k+1}) = \max(0, (k+1 - i_{k+1})Z_A - i_{k+1}Z_B),$$

$$v_B(i_{k+1}, (k+1) - i_{k+1}) = \max(0, i_{k+1}Z_B - (k+1 - i_{k+1})Z_A). \tag{11}$$

For $i < i_k$, the component contest at $(i, (k+1) - i)$ leads to $(i, k - i)$ or $(i - 1, k - (i - 1))$. As $(i_k, k - i_k)$ is a separating state, by Definition 1, the continuation values are $v_A = Z_A$ and $v_B = 0$ for both these states.¹⁸ This makes the prize of winning the component contest at $(i, (k+1) - i)$ the same for both contestants and equal to $z_A = z_B = \Delta$. Invoking Proposition 1, if $\Delta > 0$, each contestant wins this component contest with $p_A = p_B = 1/2$ and chooses expected effort $Ea = Eb = \Delta/2$. If $\Delta = 0$, $a = b = 0$ and $p_A = 1$. Accordingly, the continuation values for both contestants at state $(i, (k+1) - i)$ are the same as in $(i, k - i)$ or in $(i - 1, k - (i - 1))$:

$$v_A(i, (k+1) - i) = Z_A \quad \text{and} \quad v_B(i, (k+1) - i) = 0 \quad \text{for all } i < i_k. \tag{12}$$

For $i > i_k + 1$, the component contest at $(i, (k+1) - i)$ leads to $(i, k - i)$ if B wins and to $(i - 1, (k+1) - i)$ if A wins, with $v_A(i - 1, (k+1) - i) = v_A(i, k - i) = 0$ and $v_B(i - 1, (k+1) - i) = v_B(i, k - i) = Z_B$. If $(i, (k+1) - i)$ is already terminal, we must have $v_A = 0$ and $v_B = Z_B$. Hence, using subgame perfection, the component contest at $(i, (k+1) - i)$ is a symmetric all-pay auction with complete information. If $\Delta > 0$ equilibrium win probabilities are $p_A = p_B = 1/2$ and equilibrium expected efforts are $Ea = Eb = \Delta/2$. If $\Delta = 0$, $a = b = 0$ and $p_B = 1$. This yields

$$v_A(i, (k+1) - i) = 0 \quad \text{and} \quad v_B(i, (k+1) - i) = Z_B \quad \text{for all } i > i_k + 1. \tag{13}$$

Two states in $\Sigma(k+1)$ remain to be considered: $(i_k, k+1 - i_k)$ and $(i_k + 1, (k+1) - (i_k + 1))$. Which of them is a separating state i_{k+1} will depend on the size of $(k - i_k)Z_A - i_kZ_B$.

Let $(k - i_k)Z_A - i_kZ_B \geq 0$. As depicted in Fig. 6, by (9) this implies

$$v_A(i_k, k - i_k) = (k - i_k)Z_A - i_kZ_B \geq 0 \quad \text{and} \quad v_B(i_k, k - i_k) = 0. \tag{14}$$

From $(i_k, (k+1) - i_k)$, the state moves to $(i_k, k - i_k)$, with continuation values given in (14), or to $(i_k - 1, k - (i_k - 1))$ at which $v_A = Z_A$ and $v_B = 0$. A attributes a prize to winning at $(i_k, (k+1) - i_k)$ that is equal to $Z_A - [(k - i_k)Z_A - i_kZ_B] + \Delta$ and is at least as large as Δ by the first line of (8), and B attributes a prize to winning that is equal to Δ . If $\Delta > 0$, applying Proposition 1, both contestants randomize on the interval $[0, \Delta]$, and B does not have a mass point at Δ . As $a = \Delta$ is in A 's equilibrium support, A 's expected payoff equals the payoff from choosing $a = \Delta$, by which A wins with probability 1 the intermediate prize Δ and enters into state $(i_k - 1, k - (i_k - 1))$ at which A has a continuation value $v_A = Z_A$. Equilibrium effort Δ and the intermediate prize Δ cancel out in the payoff,

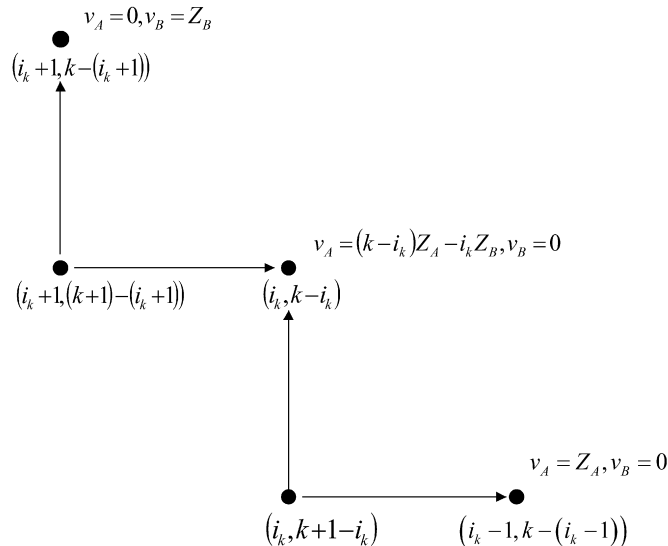


Fig. 6. Separating states in $\Sigma(k+1)$ if $v_B(i_k, k - i_k) = 0$.

¹⁸ If $i_k = 1$, then, $(i, (k+1) - i)$ is a terminal node where by (1), $v_A = Z_A$ and $v_B = 0$.

and, therefore, A 's continuation value at $(i_k, (k + 1) - i_k)$ is $v_A(i_k, (k + 1) - i_k) = v_A(i_k - 1, k - (i_k - 1)) = Z_A$. Contestant B moves from $(i_k, (k + 1) - i_k)$ to a state in which B 's continuation value is zero. Among B 's equilibrium effort choices is $b = 0$, and, as A has no mass point at $a = 0$, B loses with probability 1 when choosing $b = 0$ and moves to $(i_k - 1, k - (i_k - 1))$ with $v_B(i_k - 1, k - (i_k - 1)) = 0$. Hence, also $v_B(i_k, (k + 1) - i_k) = 0$. If $\Delta = 0$, B 's prize from winning is equal to zero, and A 's is at least zero. In equilibrium $(a, b) = (0, 0)$, and our tie-breaking rule implies A wins, the state evolves to $(i_k - 1, k - (i_k - 1))$, $v_A(i_k, k + 1 - i_k) = Z_A$, and $v_B(i_k, k + 1 - i_k) = 0$.

From $(i_k + 1, (k + 1) - (i_k + 1))$, if A wins, the players move to $(i_k, k - i_k)$ with continuation values as in (14). Otherwise, they move to $(i_k + 1, k - (i_k + 1))$, with the continuation values $v_A = 0$ and $v_B = Z_B$ by the separation property of i_k . We need to distinguish between two subcases.

Subcase 1: Let $Z_B > (k - i_k)Z_A - i_k Z_B$. Then B has a higher prize of winning than A . Making use of Proposition 1, $v_A(i_k + 1, (k + 1) - (i_k + 1)) = 0$ and $v_B(i_k + 1, (k + 1) - (i_k + 1)) = [Z_B + \Delta] - [(k - i_k)Z_A - i_k Z_B + \Delta] = (i_k + 1)Z_B - ((k + 1) - (i_k + 1))Z_A$. Note further that this value is positive, but no larger than Z_B . Hence, this v_B fulfills the conditions in the second line of (10) and of (11).

Subcase 2: Let $Z_B \leq (k - i_k)Z_A - i_k Z_B$. A has at least as large a prize of winning as B . This yields continuation values at $(i_k + 1, (k + 1) - (i_k + 1))$ of $v_B = 0$ and $v_A = [(k - i_k)Z_A - i_k Z_B + \Delta] - [Z_B + \Delta] = ((k + 1) - (i_k + 1))Z_A - (i_k + 1)Z_B$. Moreover, this $v_A \leq Z_A$. Hence, the conditions in the first lines of (10) and (11) are fulfilled.

Together with the properties of states in $\Sigma(k + 1)$ with $i < i_k$ and $i > i_k + 1$ this shows that, for the case $(k - i_k)Z_A - i_k Z_B \geq 0$ at i_k , $(i_k + 1, (k + 1) - (i_k + 1))$ is a separating state.

Turn now to the case $i_k Z_B - (k - i_k)Z_A \geq 0$. This case implies that

$$v_A(i_k, k - i_k) = 0 \quad \text{and} \quad v_B(i_k, k - i_k) = i_k Z_B - (k - i_k)Z_A. \tag{15}$$

As shown in Fig. 7, starting in $(i_k, (k + 1) - i_k)$, the state moves either to $(i_k, k - i_k)$, with continuation values given in (15), or to $(i_k - 1, k - (i_k - 1))$ at which $v_A = Z_A$ and $v_B = 0$. A attributes a prize to winning at $(i_k, (k + 1) - i_k)$ that is equal to $Z_A + \Delta$, and B attributes a prize to winning that is equal to $i_k Z_B - (k - i_k)Z_A + \Delta$. Using property (8) and $Z_B \leq Z_A$, we get $i_k Z_B - (k - i_k)Z_A \leq Z_A$. Applying Proposition 1, the equilibrium payoff is 0 for B and $(Z_A + \Delta) - [i_k Z_B - (k - i_k)Z_A + \Delta] = ((k + 1) - i_k)Z_A - i_k Z_B$ for A with $Z_A \geq ((k + 1) - i_k)Z_A - i_k Z_B \geq 0$. Moreover, starting in $(i_k + 1, (k + 1) - (i_k + 1))$, the state moves either to $(i_k, k - i_k)$, with v_A and v_B given in (15), or to $(i_k + 1, k - (i_k + 1))$ at which $v_A = 0$ and $v_B = Z_B$ by the separating property of i_k . Hence, $v_A(i_k + 1, (k + 1) - (i_k + 1)) = 0$. If $\Delta > 0$, since $b = \Delta$ is in B 's equilibrium support, makes B win with probability 1 and leads to state $(i_k + 1, k - (i_k + 1))$, B 's continuation value at $(i_k + 1, (k + 1) - (i_k + 1))$ is $v_B = Z_B$. If $\Delta = 0$, A 's prize from winning is 0 and B 's is at least zero. In equilibrium $(a, b) = (0, 0)$ and the tie-breaking rule implies B wins. Hence, $v_B = Z_B$ and $v_A = 0$.

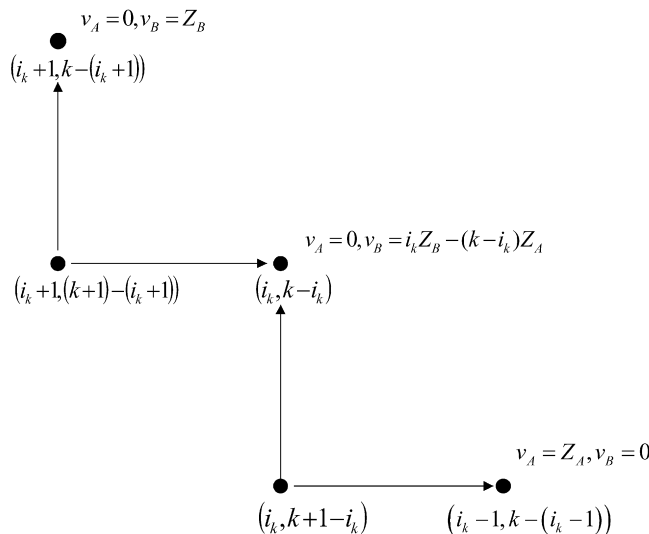


Fig. 7. Separating states in $\Sigma(k + 1)$ if $v_A(i_k, k - i_k) = 0$.

Together with the properties of states $i < i_k$ and $i > i_k + 1$ in $\Sigma(k)$ this shows that, if $i_k Z_B - (k - i_k) Z_A \geq 0$ holds, $i_{k+1} = i_k$, so that $(i_k, (k + 1) - i_k)$ is a separating state in $\Sigma(k + 1)$. Players have continuation values $v_B = 0$ and $v_A = ((k + 1) - i_{k+1}) Z_A - i_{k+1} Z_B \leq Z_A$ at this state, in line with (10) and (11).¹⁹

Overall we have shown: if $\Sigma(k)$ has a separating state then so does $\Sigma(k + 1)$, and, together with the existence of a separating state in $\Sigma(2)$ this concludes the induction proof. We now turn to the properties in Proposition 2.

For (i) recall that there cannot be more than two separating states in $\Sigma(k)$. The lemma establishes existence of at least one separating state, and together these results establish (i).

For (ii) note that (8) is equivalent to

$$\frac{j - 1}{i} \leq \frac{Z_B}{Z_A} \leq \frac{j}{i - 1} \tag{16}$$

for $i + j = k$. Hence, all separating states $(i_k, k - i_k)$ that have been constructed in the induction proof fulfill (8) and, hence fulfill (16). Suppose there is some other separating state (i, j) in $\Sigma(k)$ that does not fulfill (16). Then, by our proof of (i), there is another separating state $(i_k, k - i_k)$ in $\Sigma(k)$ that fulfills (16). If this $(i_k, k - i_k)$ fulfills both inequalities in (16) strictly, then, by (9), $v_A(i_k, k - i_k) \neq Z_A$ and $v_B(i_k, k - i_k) \neq Z_B$. This rules out that, in addition to $(i_k, k - i_k)$ a second separating state can exist. Suppose then that one of the weak inequalities holds with equality, for instance, $\frac{k - i_k - 1}{i_k} = \frac{Z_B}{Z_A}$. Using this in (9) yields $v_A(i_k, k - i_k) = Z_A$ and $v_B(i_k, k - i_k) = 0$. In turn, this implies that the only possible further separating state that may exist is $(i_k + 1, k - (i_k + 1))$. However, for this state the two inequalities in condition (16) hold: indeed, the right-hand side in (16) becomes $\frac{Z_B}{Z_A} \leq \frac{k - (i_k + 1)}{i_k + 1 - 1}$, or $(k - (i_k + 1)) Z_A \geq i_k Z_B$, which, by $\frac{k - i_k - 1}{i_k} = \frac{Z_B}{Z_A}$ is just fulfilled with equality. This shows that all separating states must fulfill the condition (16).

For the if-part of (ii), note that, for each k , the condition (16) determines either one state $(i_k, k - i_k)$ for which it holds with strict inequalities or two states such that, for one state, left-hand side inequality holds with strict equality and, for the other, the right-hand side inequality holds with strict inequality. A separating state which fulfills (16) exists from our proof of (i). If there is only one state that fulfills (16), then the separating state must be this state. If the condition determines two states, one of them must be a separating state, for which also (9) applies. But then, by the continuation values at this separating state, the other state that fulfills the condition (16) also becomes a separating state.

Turn now to (iv). The claim for separating states in (iv) holds by (8). The continuation values in non-separating states are also determined by the separating property:

$$v_A(i, j) = \begin{cases} Z_A & \text{for } i < \min[i_k: i_k \in S(k)], \\ 0 & \text{for } i > \max[i_k: i_k \in S(k)] \end{cases}$$

and

$$v_B(i, j) = \begin{cases} 0 & \text{for } i < \min[i_k: i_k \in S(k)], \\ Z_B & \text{for } i > \max[i_k: i_k \in S(k)] \end{cases}$$

with $S(k)$ the set of separating states $(i, k - i)$ in $\Sigma(k)$. The general representation of continuation values of non-separating states is then confirmed by $\min[Z_A, \max(0, (k - i) Z_A - i Z_B)] = Z_A$ as $\frac{k - i - 1}{i} > \frac{Z_B}{Z_A}$, $\min[Z_B, \max(0, i Z_B - (k - i) Z_A)] = 0$ as $\frac{k - i - 1}{i} > \frac{Z_B}{Z_A}$, $\min[Z_B, \max(0, i Z_B - (k - i) Z_A)] = Z_B$ as $\frac{Z_B}{Z_A} > \frac{k - i}{i - 1}$, and $\min[Z_A, \max(0, (k - i) Z_A - i Z_B)] = 0$ as $\frac{Z_B}{Z_A} > \frac{k - i}{i - 1}$. This completes (iv). Property (v) follows immediately from (iv).

For part (iii), consider three states, $(i, j) \in \Sigma(k - 1)$ and $(i + 1, j), (i, j + 1) \in \Sigma(k)$. Let $\frac{Z_B}{Z_A} = \frac{j}{i}$. Then by (iv), $v_A(i, j) = v_B(i, j) = 0$. By (ii), this state (i, j) has to be a separating state. In turn, using the separating property for (i, j) , it must hold that $v_A(i - 1, j + 1) = Z_A$, $v_A(i + 1, j - 1) = 0$, $v_B(i - 1, j + 1) = 0$ and $v_B(i + 1, j - 1) = Z_B$. Using the results from Proposition 1 for states $(i + 1, j)$ and $(i, j + 1)$, this implies that $v_A(i + 1, j) = 0$, $v_B(i + 1, j) = Z_B$, $v_A(i, j + 1) = Z_A$, $v_B(i, j + 1) = 0$. Accordingly, the only candidates for separating states in $\Sigma(k)$ are $(i + 1, j)$ and $(i, j + 1)$, and by existence of such a state, both must be separating states. Conversely, let $(i + 1, j)$ and $(i, j + 1)$ be separating states in $\Sigma(k)$. Then, by (8) this implies that $(j + 1) Z_A - i Z_B \leq Z_A$ and $(i + 1) Z_B - j Z_A \leq Z_B$.

¹⁹ For completeness note that we treated the case $(k - i_k) Z_A - i_k Z_B = 0$ twice. Indeed, for $(k - i_k) Z_A - i_k Z_B = 0$, both the states $(i_k + 1, (k + 1) - (i_k + 1))$ and $(i_k, (k + 1) - i_k)$ have the separating property and the induction argument from $k + 1$ to $k + 2$ works for any of these two separating states and leads to $(i_k + 1, (k + 2) - (i_k + 1))$ as a unique separating state for $k + 2$.

Both together imply that $jZ_A - iZ_B = 0$, or $\frac{Z_B}{Z_A} = \frac{j}{i}$. But by (ii) this implies that (i, j) is a unique separating state in $\Sigma(k - 1)$.

Appendix B. Local strategies for $\Delta = 0$

For $\Delta = 0$, the local strategies that are employed in states in which non-trivial component contests arise are characterized as follows.

Corollary 1 states that a non-trivial component contest arises at (i, j) if and only if $\frac{j-1}{i} < \frac{Z_B}{Z_A} < \frac{j}{i-1}$. At (i, j) the contest is an all-pay auction with prizes $z_A(i, j) = v_A(i - 1, j) - v_A(i, j - 1)$ and $z_B(i, j) = v_B(i, j - 1) - v_B(i - 1, j)$. Utilizing the characterization of equilibrium continuation values in Proposition 2 it is straightforward to show that $z_A(i, j) \geq z_B(i, j)$ as $\frac{Z_B}{Z_A} \leq \frac{j}{i}$ and, therefore, that $v_A(i, j) > 0$ if and only if $\frac{Z_B}{Z_A} < \frac{j}{i}$ and $v_B(i, j) > 0$ if and only if $\frac{Z_B}{Z_A} > \frac{j}{i}$. Moreover, the functional forms of the equilibrium distributions vary across three cases:

Case 1: Suppose $\frac{j-1}{i} < \frac{Z_B}{Z_A} \leq \frac{j-1}{i-1}$. Since $Z_B \leq Z_A$, it follows that $j - 1 < i$, which imply $j \leq i$, as j and i are both integers. Consequently, $\frac{j-1}{i-1} \leq \frac{j}{i}$ and $\frac{j-1}{i} < \frac{Z_B}{Z_A} \leq \frac{j-1}{i-1}$ implies $\frac{Z_B}{Z_A} \leq \frac{j}{i}$. In this case $z_A(i, j) = v_A(i - 1, j) - v_A(i, j - 1) = Z_A$ and $z_B(i, j) = v_B(i, j - 1) - v_B(i - 1, j) = iZ_B - (j - 1)Z_A$ so that $z_A(i, j) - z_B(i, j) = jZ_A - iZ_B \geq 0$. From Proposition 1, equilibrium distributions in state (i, j) therefore have a common support on the interval $[0, iZ_B - (j - 1)Z_A]$ and take the form $F_A(a) = \frac{a}{iZ_B - (j-1)Z_A}$ and $G_B(b) = \frac{jZ_A - iZ_B}{Z_A} + \frac{b}{Z_A}$ over that interval.

Case 2: Suppose $\frac{j-1}{i-1} < \frac{Z_B}{Z_A} \leq \frac{j}{i}$. Then $z_A(i, j) = jZ_A - (i - 1)Z_B$ and $z_B(i, j) = Z_B$, so that $z_A(i, j) - z_B(i, j) = jZ_A - iZ_B \geq 0$. From Proposition 1, equilibrium distributions in state (i, j) therefore have a common support on the interval $[0, Z_B]$ and take the form $F_A(a) = \frac{a}{Z_B}$ and $G_B(b) = \frac{jZ_A - iZ_B}{jZ_A - (i-1)Z_B} + \frac{b}{jZ_A - (i-1)Z_B}$ over that interval.

Case 3: Suppose $\frac{j}{i} \leq \frac{Z_B}{Z_A} < \frac{j}{i-1}$. Then $z_A(i, j) = jZ_A - (i - 1)Z_B$ and $z_B(i, j) = Z_B$, which implies that $z_A(i, j) - z_B(i, j) = jZ_A - iZ_B \leq 0$. From Proposition 1, equilibrium distributions in state (i, j) therefore have a common support on the interval $[0, jZ_A - (i - 1)Z_B]$ and take the form $F_A(a) = \frac{iZ_B - jZ_A}{Z_B} + \frac{a}{Z_B}$ and $G_B(b) = \frac{b}{jZ_A - (i-1)Z_B}$ over that interval.

Proposition 1 also provides simple formulas for calculating the probability that each contestant wins the component contest at (i, j) and the expected sum of efforts in the component contest. In adapting Cases 1 and 2 to these formulas, one need only insert the expressions for $z_A(i, j)$ and $z_B(i, j)$ in place of z_A and z_B in the proposition, since in both cases (and in the proposition) contestant A has the (weakly) larger prize. To apply the formulas in Proposition 1 to Case 3, all indices must be inverted since $z_A(i, j) \leq z_B(i, j)$ in Case 3, but the proposition assumes the reverse inequality.

Appendix C. Least upper bound on dissipation

Suppose $\Delta = 0$ and $Z_B = \phi Z_A$ where $1 > \phi > (n - 1)/n$. At (m, m) , $n \geq m \geq 2$, $z_A(m, m) = Z_A$, $z_B(m, m) = mZ_B - (m - 1)Z_A = Z_A[1 - (1 - \phi)m]$, so that $z_A(m, m) > z_B(m, m)$ and, from Proposition 1, the probability that B wins at (m, m) is $p_B(m, m) = \frac{[1 - (1 - \phi)m]}{2}$. In contests starting from states of the form $(m, m - 1)$, $n \geq m \geq 2$, the corresponding prizes are $z_A(m, m - 1) = (m - 1)(Z_A - Z_B) = (m - 1)(1 - \phi)Z_A$, and $z_B(m, m - 1) = Z_B = \phi Z_A$, so that $z_B(m, m - 1) - z_A(m, m - 1) = [1 - m + \phi m]Z_A$. Since by assumption $\phi > \frac{n-1}{n}$, and $\frac{n-1}{n} > \frac{m-1}{m}$ for $m < n$, it follows that $z_B(m, m - 1) > z_A(m, m - 1)$ and the probability that A wins at $(m, m - 1)$ is $p_A(m, m - 1) = \frac{(m-1)(1-\phi)}{2\phi}$. For ϕ strictly less than but close to 1 the win probability $p_A(m, m - 1)$ is positive but close to zero and $p_B(m, m)$ is close to $\frac{1}{2}$. Hence, the probability of reaching the decisive state $(1, 1)$ is positive, but can be quite small for large n .²⁰ However, the dissipation in the event that the contest reaches the decisive state can be quite large. To reach the decisive state $(1, 1)$ from the initial state (n, n) , $2n - 2$ non-trivial contests must be fought. For each non-trivial component contest (i, j) fought we know from Proposition 1 that the upper bound of the support of both contestants' equilibrium (local) strategies is the value of the smallest prize, $\min(z_A(i, j), z_B(i, j))$. Hence, it is possible for both players to draw realizations of effort arbitrarily close to this upper bound in each component contest. For $C^{m,m}$ this upper bound

²⁰ We have derived an exact expression for this probability, but it is not very enlightening. A very loose upper bound on the probability, is $2^{-2(n-1)}$, which would arise if the player with the smaller prize value at each stage of the form (m, m) or $(m, m - 1)$ won with probability $\frac{1}{2}$.

is $Z_A[1 - (1 - \phi)m]$ and for $C^{m,m-1}$ it is $Z_A(m - 1)(1 - \phi)$. Hence, for ϕ very close to 1 the supremum of the support of each player's equilibrium effort distribution is close to zero in $C^{m,m-1}$ but approaches Z_A in $C^{m,m}$. If realizations of the local strategies occur arbitrarily close to this upper bound of the equilibrium support for each player in each component contest on the path from the state (n, n) to state $(1, 1)$, and again in the decisive contest at $(1, 1)$, the total effort expended by a single contestant could reach arbitrarily close to nZ_A . Obviously, realizations of aggregate overdissipation could be double this.

Appendix D. Characterization of equilibrium for $Z_A = Z_B = Z, \Delta > 0$

With $Z_A = Z_B \equiv Z$ and $\Delta > 0$, from Proposition 2 we know that (i, j) is a separating state if and only if $\frac{j-1}{i} \leq \frac{Z_B}{Z_A} = 1 \leq \frac{j}{i-1}$. An immediate consequence is that if $(i, j) \in \Sigma(2k)$ for $k \geq 1$, then $(i, j) = (k, k)$ is the unique separating state in $\Sigma(2k)$. If $(i, j) \in \Sigma(2k + 1)$ for $k \geq 1$, then $(i, j) = (k, k + 1)$ and $(i, j) = (k + 1, k)$ comprise the set of separating states. In the former case it is straightforward to show, again from Proposition 2, that $z_A(k, k) = v_A(k - 1, k) - v_A(k, k - 1) + \Delta = Z + \Delta$ and $z_B(k, k) = v_B(k, k - 1) - v_B(k - 1, k) + \Delta = Z + \Delta$, so that, Proposition 1 then implies that a symmetric non-trivial all-pay auction is played at $(i, j) = (k, k)$. This auction dissipates all rents, so that $v_A(k, k) = v_B(k, k) = 0$, $E(a + b) = Z + \Delta$, and, from the symmetry of the component contest, we find that each contestant is equally likely to win: $p_A = p_B = \frac{1}{2}$. This then immediately allows us to derive the expected values of the prizes at the component contests starting in states of the form $(i, j) = (k, k + 1)$ and $(i, j) = (k + 1, k)$. In the former case, $z_A(k, k + 1) = v_A(k - 1, k + 1) - v_A(k, k) + \Delta = Z + \Delta$ and $z_B(k, k + 1) = v_B(k, k) - v_B(k - 1, k + 1) + \Delta = \Delta$. Hence, from Proposition 1 at $(i, j) = (k, k + 1)$ we have an asymmetric component contest with $v_A(k, k + 1) = Z$, $v_B(k, k + 1) = 0$, $E(a + b) = \frac{\Delta}{2} + \frac{\Delta}{2}(\frac{\Delta}{Z + \Delta})$, $p_A = 1 - \frac{\Delta}{2(Z + \Delta)}$ and $p_B = \frac{\Delta}{2(Z + \Delta)}$. In a similar fashion, it is straightforward to show that at $(i, j) = (k + 1, k)$, $z_A(k + 1, k) = \Delta$, $z_B(k + 1, k) = Z + \Delta$, $v_A(k + 1, k) = 0$, $v_B(k + 1, k) = Z$, $E(a + b) = \frac{\Delta}{2} + \frac{\Delta}{2}(\frac{\Delta}{Z + \Delta})$, $p_B = 1 - \frac{\Delta}{2(Z + \Delta)}$ and $p_A = \frac{\Delta}{2(Z + \Delta)}$.

To calculate expected total effort and component contest win probabilities for states that are not separating, we divide up the analysis. Suppose first that $(i, j) \in \Sigma(2k)$ for $k \geq 1$, but that $i < k$ (for $k = 1$ this set is empty). Then (i, j) is not a separating state and we claim that in the component contest starting in (i, j) , $E(a + b) = \Delta$ and $p_A = p_B = \frac{1}{2}$. To see this, we examine the two states $(i - 1, j)$ and $(i, j - 1)$ that can be immediately reached from the component contest at (i, j) . If contestant A wins the component contest, the state moves to $(i - 1, j)$. Note that $(i - 1, j)$ is either a winning terminal state for player A or $(i - 1, j) \in \Sigma(2k - 1)$ with $i - 1 < k - 1$. In either case, $v_A(i - 1, j) = Z$ and $v_B(i - 1, j) = 0$. If contestant B wins the contest at (i, j) the state moves to $(i, j - 1) \in \Sigma(2k - 1)$ with $i \leq k - 1$, so again $v_A(i, j - 1) = Z$ and $v_B(i, j - 1) = 0$. (Note that if $i = k - 1$, $(i, j - 1) \in \Sigma(2k - 1)$ is of the form $(k - 1, k)$ and, even though this is a separating state, it still satisfies $v_A(k - 1, k) = Z$ and $v_B(k - 1, k) = 0$). Hence, the prizes contested in the component contest at (i, j) are $z_A(i, j) = \Delta$ and $z_B(i, j) = \Delta$. From Proposition 1 this yields $E(a + b) = \Delta$ and $p_A = p_B = \frac{1}{2}$. A similar argument applies to the case where $(i, j) \in \Sigma(2k)$ for $k \geq 1$, but that $i > k$. Hence, as is the case for the state (k, k) , for any off-diagonal state $(i, j) \in \Sigma(2k)$, $p_A = p_B = \frac{1}{2}$. However, the expected aggregate effort in an off-diagonal state $(i, j) \in \Sigma(2k)$ is $E(a + b) = \Delta$.

A somewhat more straightforward argument shows that the same type of result holds for non-separating states contained in $\Sigma(2k + 1)$. Suppose that $(i, j) \in \Sigma(2k + 1)$ for $k \geq 1$, and (i, j) is not a separating state. Then, as we have just shown, any state that may be reached immediately from the component contest at (i, j) is either a terminal state or an element of $\Sigma(2k)$ for which the advantaged contestant has continuation value Z and the disadvantaged player has continuation value 0. Hence the prize values for both contestants at (i, j) are equal to Δ and Proposition 1 demonstrates that this rent is completely dissipated and each player is equally likely to win the component contest. That is, at (i, j) the expected aggregate effort is $E(a + b) = \Delta$ and $p_A = p_B = \frac{1}{2}$.

References

Baye, M.R., Kovenock, D., de Vries, C., 1994. The solution to the Tullock rent-seeking game when $R > 2$: Mixed strategy equilibria and mean dissipation rates. *Public Choice* 81 (3-4), 363-380.
 Baye, M.R., Kovenock, D., de Vries, C., 1996. The all-pay auction with complete information. *Econ. Theory* 8 (2), 291-305.
 Baye, M.R., Kovenock, D., de Vries, C., 1999. The incidence of overdissipation in rent-seeking contests. *Public Choice* 99 (3-4), 439-454.

- Budd, C., Harris, C., Vickers, J., 1993. A model of the evolution of duopoly: Does the asymmetry between firms tend to increase or decrease? *Rev. Econ. Stud.* 60 (3), 543–573.
- Che, Y.-K., Gale, I., 2000. Difference-form contests and the robustness of all-pay auctions. *Games Econ. Behav.* 30 (1), 22–43.
- Clark, D.J., Riis, C., 2000. Allocation efficiency in a competitive bribery game. *J. Econ. Behav. Organ.* 42 (1), 109–124.
- Dasgupta, P., 1986. The theory of technological competition. In: Stiglitz, J.E., Mathewson, G.F. (Eds.), *New Developments in the Analysis of Market Structure*. MIT Press, Cambridge, pp. 519–547.
- Ellingsen, T., 1991. Strategic buyers and the social cost of monopoly. *Amer. Econ. Rev.* 81 (3), 648–657.
- Fudenberg, D., Gilbert, R., Stiglitz, J.E., Tirole, J., 1983. Preemption, leapfrogging and competition in patent races. *Europ. Econ. Rev.* 22 (1), 3–31.
- Harris, C., Vickers, J., 1985. Perfect equilibrium in a model of a race. *Rev. Econ. Stud.* 52 (2), 193–209.
- Harris, C., Vickers, J., 1987. Racing with uncertainty. *Rev. Econ. Stud.* 54 (1), 1–21.
- Hillman, A.L., Riley, J.G., 1989. Politically contestable rents and transfers. *Econ. Politics* 1, 17–40.
- Hoehn, T., Szymanski, S., 1999. The Americanization of European football. *Econ. Policy* 14 (28), 203–240.
- Hörner, J., Sahuguet, N., 2004. A war of attrition with endogenous effort levels. Unpublished manuscript. Northwestern University.
- Klump, T., Polborn, M.K., 2006. Primaries and the New Hampshire effect. *J. Public Econ.* 90 (6–7), 1073–1114.
- Kornhauser, L., Rubinstein, A., Wilson, C., 1989. Reputation and patience in the ‘war of attrition’. *Economica* 56 (221), 15–24.
- Lazear, E.P., Rosen, S., 1981. Rank-order tournaments as optimum labor contracts. *J. Polit. Economy* 89 (5), 841–864.
- Lee, T., Wilde, L.L., 1980. Market structure and innovation: A reformulation. *Quart. J. Econ.* 94 (2), 429–436.
- Leininger, W., 1991. Patent competition, rent dissipation, and the persistence of monopoly: The role of research budgets. *J. Econ. Theory* 53 (1), 146–172.
- Malueg, D.A., Yates, A.J., 2006. Best-of-three contests between equally-skilled players. Unpublished manuscript.
- Moldovanu, B., Sela, A., 2001. The optimal allocation of prizes in contests. *Amer. Econ. Rev.* 91 (3), 542–558.
- Moldovanu, B., Sela, A., 2006. Contest architecture. *J. Econ. Theory* 126 (1), 70–96.
- Nalebuff, B.J., Stiglitz, J.E., 1983. Prizes and incentives: Towards a general theory of compensation and competition. *Bell J. Econ.* 14 (1), 21–43.
- Szymanski, S., 2003. The economic design of sporting contests. *J. Econ. Lit.* 41 (4), 1137–1187.
- Tullock, G., 1980. Efficient rent seeking. In: Buchanan, J., Tollison, R.D., Tullock, G. (Eds.), *Towards a Theory of the Rent-Seeking Society*. Texas A&M Univ. Press, College Station, pp. 97–112.